Complex Integrals and the Laplace Transform

There are a few computations for which the complex exponential is very nice to use. We'll see a few here, but first we show that integrating the complex exponential works out the same as the real exponential:

Theorem: As in the real case, $\int \mathrm{e}^{at}\,dt = \frac{1}{a}\mathrm{e}^{at}$, we have: $\int \mathrm{e}^{(bi)t}\,dt = \frac{1}{bi}\mathrm{e}^{(bi)t}$

Proof:

$$\int \mathrm{e}^{(bi)t}\,dt = \int \mathrm{e}^{(bt)i}\,dt = \int \cos(bt) + i\sin(bt)\,dt = \int \cos(bt)\,dt + i\int \sin(bt)\,dt =$$

$$\frac{1}{b}\sin(bt) - \frac{i}{b}\cos(bt) = \frac{\sin(bt) - i\cos(bt)}{b}$$

And

$$\frac{1}{bi}\mathrm{e}^{(bt)i} = \frac{\cos(bt) + i\sin(bt)}{bi} \cdot \frac{i}{i} = \frac{-\sin(bt) + i\cos(bt)}{-b} = \frac{\sin(bt) - i\cos(bt)}{b}$$

Therefore, these quantities are the same.

Theorem: As in the real case the previous theorem works for a full complex exponential: $\int \mathrm{e}^{(a+bi)t}\,dt = \frac{1}{(a+bi)}\mathrm{e}^{(a+bi)t}$

You can work this out, but it is more complicated since we'll need to do integration by parts twice for each integral. It is a nice exercise to try out when you have a little time.

Theorem: As in the proof of the last theorem, a corollary is the following:

$$\int \mathrm{e}^{at}\cos(bt)\,dt = \mathrm{Re}\left(\int \mathrm{e}^{(a+bi)t}\,dt\right) = \mathrm{Re}\left(\frac{1}{a+ib}\mathrm{e}^{(a+ib)t}\right)$$

$$\int \mathrm{e}^{at}\sin(bt)\,dt = \mathrm{Im}\left(\int \mathrm{e}^{(a+bi)t}\,dt\right) = \mathrm{Im}\left(\frac{1}{a+ib}\mathrm{e}^{(a+ib)t}\right)$$

Proof: The proof just uses the definition and the earlier theorem. Here we show the proof for the cosine; the proof for the sine is very similar. By Euler's Formula:

$$\mathrm{e}^{at}\cos(bt) + i\mathrm{e}^{at}\sin(bt) = \mathrm{e}^{(a+ib)t}$$

Therefore, we can integrate both sides:

$$\int \mathrm{e}^{at}\cos(bt)\,dt + i\int \mathrm{e}^{at}\sin(bt)\,dt = \int \mathrm{e}^{(a+ib)t}\,dt = \frac{1}{a+ib}\mathrm{e}^{(a+ib)t}$$

Now, take the real part of each expression:

$$\int \mathrm{e}^{at}\cos(bt)\,dt = \mathrm{Re}\left(\int \mathrm{e}^{(a+ib)t}\,dt\right) = \mathrm{Re}\left(\frac{1}{a+ib}\mathrm{e}^{(a+ib)t}\right)$$

Of course, after this we need to actually compute this last expression. Recall that division by complex numbers is defined as:

$$\frac{1}{a+ib} = \frac{1}{a+ib} \cdot \frac{a-ib}{a-ib} = \frac{a}{a^2+b^2} - i\,\frac{b}{a^2+b^2}$$

Worked Example:

1. Use complex exponentials to compute $\int e^{2t}\cos(3t)\,dt$.

   SOLUTION: We note that $e^{2t}\cos(3t) = \mathrm{Re}(e^{(2+3i)t})$, so:

$$\int e^{2t}\cos(3t)\,dt = \mathrm{Re}\left(\frac{1}{2+3i}e^{(2+3i)t}\right)$$

   Simplifying the term inside the parentheses and multiplying out the complex terms:

$$e^{2t}\left(\frac{2-3i}{4+9}\right)(\cos(3t)+i\sin(3t)) =$$

$$e^{2t}\left[\left(\frac{2}{13}\cos(3t)+\frac{3}{13}\sin(3t)\right)+i\left(-\frac{3}{13}\cos(3t)+\frac{2}{13}\sin(3t)\right)\right]$$

   Therefore,

$$\int e^{2t}\cos(3t)\,dt = e^{2t}\left(\frac{2}{13}\cos(3t)+\frac{3}{13}\sin(3t)\right)$$

   In fact, we get the other integral for free:

$$\int e^{2t}\sin(3t)\,dt = e^{2t}\left(\frac{-3}{13}\cos(3t)+\frac{2}{13}\sin(3t)\right)$$

2. Use complex exponentials to compute the Laplace transform of $\cos(at)$:

   SOLUTION: Note that $\cos(at) = \mathrm{Re}(e^{(at)i})$

$$\mathcal{L}(\cos(at)) = \int_0^\infty e^{-st}\cos(at)\,dt = \mathrm{Re}\left(\int_0^\infty e^{-st}e^{(ai)t}\,dt\right) =$$

$$\mathrm{Re}\left(\int_0^\infty e^{-(s-ai)t}\,dt\right) = \mathrm{Re}\left(\frac{-1}{(s-ai)}e^{-(s-ai)t}\bigg|_{t=0}^{t\to\infty}\right.$$

   What happens to our expression as $t\to\infty$? The easiest way to take the limit is to check the magnitude (see if it is going to zero):

$$\left|\frac{-1}{s-ai}e^{-st}e^{(ai)t}\right| = \left|\frac{-1}{s-ai}\right|\cdot\left|e^{-st}\right|\cdot\left|e^{(ai)t}\right|$$

   Now, the first term is a constant with respect to t, and $e^{(at)i}$ is a point on the unit circle (see the exercises). Therefore, the magnitude of the integral depends solely on e^{-st}, where s is any real number so that e^{-st} is the standard real exponential function. For $s>0$, the function $e^{-st}\to 0$ as $t\to\infty$, so that the entire expression will also go to zero if $s>0$:

$$\lim_{t\to\infty}\frac{-1}{(s-ai)}e^{-(s-ai)t} = 0$$

   and the Laplace transform is:

$$\mathcal{L}(\cos(at)) = \mathrm{Re}\left(0-\frac{-1}{s-ai}\right) = \mathrm{Re}\left(\frac{s+ai}{s^2+a^2}\right) = \frac{s}{s^2+a^2}$$

   As a side remark, we get the Laplace transform of $\sin(at)$ for free since it is the imaginary part.

Homework Addition to Section 6.1

1. Show that e^{iat} is on the unit circle- That is, show that its magnitude is 1 (for any value of a, t).

2. Use complex exponentials to compute $\int e^{-2t}\sin(3t)\,dt$.

3. Use complex exponentials to compute the Laplace transform of $\sin(at)$.

4. Use complex exponentials to compute the Laplace transform of $e^{at}\sin(bt)$ and $e^{at}\cos(bt)$ (compare to exercises 13, 14).

5. Prove that e^t goes to infinity faster than any polynomial. You can do that by showing

$$\lim_{t\to\infty}\frac{t^n}{e^t}=0$$

6. Use the "Racetrack Principle"[1] to prove that $\ln(t) < t$ for all $t > 1$.

7. Show that, if $f(t)$ is bounded (that is, there is a constant A so that $|f(t)| \leq A$ for all t), then f is of exponential order (do this by finding K, a and M from the definition).

8. If the function is of exponential order, find the K, a and M from the definition. Otherwise, state that it is not of exponential order.

   Something that may be handy from algebra: $A = e^{\ln(A)}$.

   (a) $\sin(t)$
   (b) $\tan(t)$
   (c) t^3
   (d) e^{t^2}
   (e) 5^t
   (f) t^t

9. Use complex exponentials to find the Laplace transform of $t\sin(at)$.

[1]The Racetrack Principle says that if $f(a) \leq g(a)$ and $f'(x) < g'(x)$ for all x in some interval (a, b), then $f(x) < g(x)$ for all x in (a, b) (think of $f(x)$ and $g(x)$ as the positions of two cars on a track, so that the derivatives are their velocities).