Solutions to: Trigonometry Review: Cosine Sums and Beats

1. For each sum of cosines, find the frequency of the envelope and the fast varying wave:

   (a) $\cos(2\pi \cdot 11.5\, t) + \cos(2\pi \cdot 12\, t)$
   Rewriting the sum as before, we get:

   $$\cos(2\pi \cdot 11.5\, t) + \cos(2\pi \cdot 12\, t) = 2\cos\left(2\pi \cdot 0.25t\right)\cos\left(2\pi \cdot 11.75\, t\right)$$

   so the frequency (not circular) of the envelope is 1/4, and it is periodic with period 4.

   (b) Similarly, we see that:

   $$\cos(2\pi \cdot 11.75\, t) + \cos(2\pi \cdot 12\, t) = 2\cos\left(2\pi \cdot 0.125t\right)\cos\left(2\pi \cdot 11.875\, t\right)$$

   so the frequency is 1/8, and it is periodic with period 8.

   (c) And finally,

   $$\cos(2\pi \cdot 11.99\, t) + \cos(2\pi \cdot 12\, t) = 2\cos\left(2\pi \cdot 0.005t\right)\cos\left(2\pi \cdot 11.995\, t\right)$$

   so the frequency is 1/200, and the period is 200.

2. In light of the previous exercise, what is happening to the frequency of the envelope as the two frequencies come together?

   SOLUTION: As the two frequencies come together, the frequency of the envelope is going to zero (making the period go to infinity).

3. Starting with our two cosine identities, show that the following identity can be obtained (this is the one that is used in our text):

   Typo on the first sine function (Corrected below)

   $$\cos(A) - \cos(B) = 2\sin\left(\frac{B-A}{2}\right)\sin\left(\frac{A+B}{2}\right)$$

   SOLUTION:

   $$\begin{array}{rcl} \cos(x-y) & = & \cos(x)\cos(y) + \sin(x)\sin(y) \\ -\cos(x+y) & = & -\cos(x)\cos(y) + \sin(x)\sin(y) \\ \hline \cos(x-y) - \cos(x+y) & = & +2\sin(x)\sin(y) \end{array}$$

   Finally, let $A = x - y$ and $B = x + y$, so that $x = \frac{A+B}{2}$ and $y = \frac{B-A}{2}$.

4. Find the limit of the expression below, as $\omega \to 12$:

   $$\frac{\cos(2\pi \cdot \omega t) - \cos(2\pi \cdot 12t)}{144 - \omega^2}$$

   SOLUTION: We can use l'Hospital's rule, since we have the "0/0" indeterminant form.

   $$\lim_{\omega \to 12} \frac{\cos(2\pi \cdot \omega t) - \cos(2\pi \cdot 12t)}{144 - \omega^2} = \lim_{\omega \to 12} \frac{-2\pi t \sin(2\pi \cdot \omega t))}{-2\omega} = \frac{\pi}{12} t\, \sin(2\pi \cdot 12t)$$

5. Are the following functions periodic? If so, find the period (or you can find the frequency instead). To make the analysis simpler, we've put 2π in each. If the function is not periodic, state why.

   (a) $\cos(2\pi \cdot 2t) + \sin(2\pi \cdot 2t)$
   This function is periodic with frequency 2 (or period 1/2).

   (b) $\cos(2\pi \cdot 2t) + \cos(2\pi \cdot 3t)$
   The first function is periodic with period 1/2, the second is periodic with period 1/3. For the sum to be periodic, the period will be the least common multiple of the two periods. Therefore, there must be integers k, m so that

   $$\frac{1}{2}k = \frac{1}{3}m \quad \Rightarrow \quad 3k = 2m$$

   So, setting $k = 2$ and $m = 3$, we see that the period of the sum is 1.

   (c) $\cos(2\pi \cdot 2t) + \cos(2\pi \cdot \sqrt{2}\,t)$
   Trying to repeat what we just did, is it possible to find positive integers k and m so that

   $$\frac{1}{2}k = \frac{1}{\sqrt{2}}m \quad \Rightarrow \quad \sqrt{2}\,k = 2m$$

   No, this is impossible since $\sqrt{2}$ is irrational (note that if there were such integers k, m, then $\sqrt{2} = 2m/k$, which is rational).

6. Standard A corresponds to a frequency of 440 Hz (or 440 cycles per second).

   (a) A tuning fork tuned to standard A makes a sound that is proportional to $\cos(\omega t)$. Find ω.
   SOLUTION: The frequency of $\cos(\omega t)$ is $\frac{\omega}{2\pi}$, so:

   $$\frac{\omega}{2\pi} = 440 \quad \Rightarrow \quad \omega = 2\pi \cdot 440$$

   (b) Suppose a piano generates a sound that is 438 Hz, and the amplitude of the sound is proportional to $\cos(\alpha t)$. Find α.
   SOLUTION: Similar to the previous problem, $\alpha = 2\pi \cdot 438$

   (c) What is the frequency of the beating that you hear when the piano string and tuning fork in the previous parts are struck at the same time?
   SOLUTION: We'll hear the frequency of the envelope. The argument of the first cosine term will be:

   $$\frac{B - A}{2} = 2\pi \cdot \frac{440 - 438}{2} = 2\pi$$

   Therefore, we will hear a beat of 1 Hz (or 1 cycle per second).

   (d) If the tuning fork and a random piano string are struck at the same time and you hear an oscillation of 5 Hz, what are the possible frequencies of the piano?
   SOLUTION: The tuning fork was 440, so if C is the frequency of the piano string, then

   $$\frac{|440 - C|}{2} = 5$$

   so the frequency can be 10 Hz above or below the tuning fork.