Finishing 3.7, 3.8:

The following example will give us an idea of what to expect when solving systems with damping and forcing. Suppose that

$$y'' + 3y' + 2y = 10\sin(t) \qquad y(0) = -1, \quad y'(0) = 1$$

Then the solution is

$$y(t) = 4\mathrm{e}^{-t} - 2\mathrm{e}^{-2t} - 3\cos(t) + \sin(t) \tag{1}$$

The qualitative properties of this solution are:

- The solution may be regarded as the sum of a transient and a steady-state function.

  The transient part of the solution is the part that goes to zero as $t \to \infty$, which is the homogeneous part of the solution. The steady state term is the particular solution.

- The frequency of the steady state solution is the same as the frequency of the forcing.

  In this case, both the forcing and steady state solutions are periodic with period 2π. In contrast to undamped oscillators with periodic forcing, there is no beating.

- The qualitative behavior of the solution (for relatively large t) is insensitive to changes in the initial conditions.

  This may be surprising, but think about which part of the general solution has the arbitrary constants that are used to solve a general IVP- It is the homogeneous part of the solution. In this case, the general solution was:

  $$y(t) = C_1\mathrm{e}^{-t} + C_2\mathrm{e}^{-2t} + y_p(t)$$

  so that the steady state solution is the particular part.

These three properties summarize our analysis of these types of system. Now we come to another key question for these systems: Is it possible to have something akin to *resonance* in a forced, damped linear system that has a periodic forcing?

Gain and Lag

The notation below is a bit different than our text, and I think it simplifies the discussion quite a bit. Consider the DE:

$$y'' + cy' + \omega^2 y = F\sin(\alpha t)$$

(so mass has been set to 1, and we'll use a sine instead of a cosine for the forcing).

Then the *steady state* solution is[1]:

$$y_{ss}(t) = \left(\frac{F}{(\omega^2-\alpha^2)^2 + \alpha^2c^2}\right)\left(\left(\omega^2-\alpha^2\right)\sin(\alpha t) - \alpha c\cos(\alpha t)\right)$$

[1] We'll prove this at the end in the exercises.

The amplitude of the steady state (written as a single sine) is then[2]:

$$A = \frac{F}{\sqrt{(\omega^2 - \alpha^2)^2 + \alpha^2 c^2}}$$

If we divide the amplitude by the amplitude of the forcing function F, we get what is called the **gain**, G:

$$G = \frac{A}{F} = \frac{1}{\sqrt{(\omega^2 - \alpha^2)^2 + \alpha^2 c^2}}$$

The maximum value for G occurs where the derivative is zero:

$$\frac{dG}{d\alpha} = -\frac{1}{2}((\omega^2 - \alpha^2)^2 + \alpha^2 c^2)^{-3/2}(2(\omega^2 - \alpha^2)(-2\alpha) + 2\alpha c^2)$$

Setting that to zero and solving, we get

$$\alpha = \frac{1}{2}\sqrt{4\omega^2 - 2c^2}$$

Therefore, the maximum amplitude of the particular solution occurs not at ω anymore, but at a value close to ω (as long as c is small). FYI, the maximum value of the gain can be computed:

$$G_{\text{max}} = \frac{2}{c\sqrt{4\omega^2 - c^2}}$$

Discussion: Why is the preceeding computation important? In the real world, we will not know the natural frequency of a system. However, if we have access to a mechanism that provides a periodic forcing (like an amplified speaker hooked up to a computer), then by adjusting the period of the forcing, we can measure the amplitude of the response (the motion of the object). For example, the object might be a wine glass (or a beaker), and the forcing is provided by a strongly amplified sound (See the video on the class website).

Example: Let $y'' + \frac{1}{10}y' + 9y = \sin(\alpha t)$ The amplitude of the forcing is $F = 1$. The gain is:

$$G = \frac{1}{\sqrt{(9 - \alpha^2)^2 + \frac{1}{100}\alpha^2}}$$

Using the formulas above, we see that, to maximize the gain (which is basically the amplitude of the response), we would set

$$\alpha = \frac{1}{2}\sqrt{4 \cdot 9 - \frac{2}{100}} = \frac{1}{2} = \frac{1}{2}\sqrt{35.98} \approx 2.9992$$

And, using that forcing frequency, we get a maximum amplitude of the response:

$$G_{\text{max}} \approx 3.335$$

What are the practical implications of what we have just learned?

- "Resonance" occurs for lightly damped systems.
- In the case of a lightly damped system, how might we learn the (approximate) natural frequency ω of a system? (Assume we can "tune" α and see the response, $u(t)$).

[2]We'll also prove this in the exercises.

Exercises:

1. Find the particular solution to:

$$y'' + cy' + \omega^2 y = F\sin(\alpha t)$$

   Hint: Use the Method of Undetermined Coefficients and then Cramer's Rule. As you go through the computations, remember that α, ω and c are fixed parameters (so your only unknowns are coming from the Undetermined Coefficients).

2. Given that the particular solution (which is the steady state solution in this case) to the previous problem is:

$$\left(\frac{F}{(\omega^2-\alpha^2)^2+\alpha^2c^2}\right)\left(\left(\omega^2-\alpha^2\right)\sin(\alpha t)-\alpha c\cos(\alpha t)\right)$$

   Find the expression for the amplitude of the steady state, using our earlier formula:

$$A\cos(\omega t) + B\sin(\omega t) = R\cos(\omega t - \delta)$$

3. Consider the following forced spring-mass system (also known as an oscillator):

$$y'' + \frac{1}{2}y' + 4y = \sin(\alpha t)$$

   In the previous problem, we showed that the amplitude of the particular solution is given by:

$$A = \frac{F}{\sqrt{(\omega^2-\alpha^2)^2+\alpha^2c^2}}$$

   (a) Use the previous formula to find the amplitude of our particular solution in terms of α:
   (b) Determine the value of α for which the amplitude is a maximum.
   (c) If the damping was set to zero, what is the (circular) frequency of the resulting homogeneous solution (which our text calls ω_0)?
   (d) With the damping back in:
      i. Find the transient part of the solution to the ODE:
      ii. While the transient part is not itself periodic, we say that it is "quasi-periodic". What is the quasi-frequency (which we'll refer to as μ)?
   (e) Verify the approximation in our text that said the following (see pg. 198): By comparing the quasi-frequency μ with ω_0, we find that (in terms of the original spring-mass equation):

$$\frac{\mu}{\omega_0} \approx 1 - \frac{\gamma^2}{8km}$$

   (f) Compare the maximizing value of α with the frequency of the undamped homogeneous DE and the pseudo frequency of the damped system. What should we find?

Extra Exercises, Sections 3.7 and 3.8