Solutions to the Exercises:

1. Find the particular solution to:

$$y'' + cy' + \omega^2 y = F\sin(\alpha t)$$

Hint: Use the Method of Undetermined Coefficients and then Cramer's Rule. As you go through the computations, remember that α, ω and c are fixed parameters (so your only unknowns are coming from the Undetermined Coefficients).

SOLUTION: Since $c \neq 0$, the particular part of the solution will always be

$$y_p = A\cos(\alpha t) + B\sin(\alpha t)$$

Substitute this into the DE and solve for A, B. Hint: It is much easier to keep track of the algebra if you line things up with the cosine and sine:

$$\begin{array}{rrlrl} \omega^2 y_p & = A\omega^2 & \cos(\alpha t) & +B\omega^2 & \sin(\alpha t) \\ c y_p' & = B\alpha c & \cos(\alpha t) & -A\alpha c & \sin(\alpha t) \\ y_p'' & = -A\alpha^2 & \cos(\alpha t) & -B\alpha^2 & \sin(\alpha t) \\ \hline F\sin(\alpha t) & = (A(\omega^2-\alpha^2)+B\alpha c) & \cos(\alpha t) & +(B(\omega^2-\alpha^2)-A\alpha c) & \sin(\alpha t) \end{array}$$

From this, we get the system of equations:

$$\begin{aligned} A(\omega^2-\alpha^2)+B\alpha c &= 0 \\ -A\alpha c + B(\omega^2-\alpha^2) &= F \end{aligned}$$

Now use Cramer's Rule to get that

$$A = \frac{F(-\alpha c)}{(\omega^2-\alpha^2)^2+\alpha^2c^2} \qquad B = \frac{F(\omega^2-\alpha^2)}{(\omega^2-\alpha^2)^2+\alpha^2c^2}$$

2. Given that the particular solution (which is the steady state solution in this case) to the previous problem is:

$$\left(\frac{F}{(\omega^2-\alpha^2)^2+\alpha^2c^2}\right)\left(\left(\omega^2-\alpha^2\right)\sin(\alpha t) - \alpha c\cos(\alpha t)\right)$$

Find the expression for the amplitude of the steady state, using our earlier formula:

$$A\cos(\omega t) + B\sin(\omega t) = R\cos(\omega t - \delta)$$

SOLUTION: The amplitude is $R = \sqrt{A^2+B^2}$, which is:

$$\sqrt{\left(\frac{F(-\alpha c)}{(\omega^2-\alpha^2)^2+\alpha^2c^2}\right)^2 + \left(\frac{F(\omega^2-\alpha^2)}{(\omega^2-\alpha^2)^2+\alpha^2c^2}\right)^2} = \sqrt{\frac{F^2[(\omega^2-\alpha^2)^2+\alpha^2c^2]}{[(\omega^2-\alpha^2)^2+\alpha^2c^2]^2}}$$

This simplifies to:

$$R = \frac{F}{\sqrt{(\omega^2-\alpha^2)^2+\alpha^2c^2}}$$

3. Consider the following forced spring-mass system (also known as an oscillator):

$$y'' + \frac{1}{2}y' + 4y = \sin(\alpha t)$$

In the previous problem, we showed that the amplitude of the particular solution is given by:

$$A = \frac{F}{\sqrt{(\omega^2 - \alpha^2)^2 + \alpha^2 c^2}}$$

(a) Use the previous formula to find the amplitude of our particular solution in terms of α:

SOLUTION: In this example, $F = 1$, $c = \frac{1}{2}$ and $\omega = 2$. Therefore,

$$A = \frac{1}{\sqrt{(4-\alpha^2)^2 + \frac{\alpha^2}{4}}} = \frac{2}{\sqrt{4(4-\alpha^2)^2 + \alpha^2}}$$

(b) Determine the value of α for which the amplitude is a maximum.

SOLUTION: Differentiate with respect to α (and set to zero):

$$\frac{dA}{d\alpha} = 2 \cdot -\frac{1}{2}\left(4(4-\alpha^2)^2 + \alpha^2\right)^{-3/2}\left(8(4-\alpha^2)(-2\alpha) + 2\alpha\right) = 0$$

Therefore,

$$-2\alpha\left[32 - 8\alpha^2 - 1\right] = 0 \quad \alpha = 0 \quad \text{or} \quad \alpha = \pm\sqrt{\frac{31}{8}}$$

And we note that this is the same as what you would get from the handout:

$$\alpha = \frac{1}{2}\sqrt{4\omega^2 - 2c^2} = \frac{1}{2}\sqrt{\frac{31}{2}} = \sqrt{\frac{31}{8}}$$

(c) If the damping was set to zero, what is the (circular) frequency of the resulting homogeneous solution (which our text calls ω_0)?

SOLUTION: With the damping set to zero, the frequency is 2.

(d) With the damping back in:

i. Find the transient part of the solution to the ODE:

SOLUTION: The roots to the characteristic equation are:

$$r = \frac{1}{2}\left(-\frac{1}{2} \pm \sqrt{\frac{1}{4} - 16}\right) = -\frac{1}{4} \pm \frac{\sqrt{63}}{4} = -\frac{1}{4} \pm \frac{3\sqrt{7}}{4}i$$

Therefore,

$$y_h(t) = \mathrm{e}^{-t/4}\left(C_1 \cos\left(\frac{3\sqrt{7}}{4}t\right) + C_2 \sin\left(\frac{3\sqrt{7}}{4}t\right)\right)$$

ii. While the transient part is not itself periodic, we say that it is "quasi-periodic". What is the quasi-frequency (which we'll refer to as μ)?
SOLUTION: The quasi-period is

$$\frac{2\pi \cdot 4}{3\sqrt{7}}$$

The quasi (circular) frequency is $\frac{3\sqrt{7}}{4} \approx 1.98$ (Notice it dropped a bit from the undamped version at 2).

(e) Verify the approximation in our text that said the following (see pg. 198): By comparing the quasi-frequency μ with ω_0, we find that (in terms of the original spring-mass equation):

$$\frac{\mu}{\omega_0} \approx 1 - \frac{\gamma^2}{8km}$$

SOLUTION: On the one hand,

$$\frac{\mu}{\omega_0} = \frac{3\sqrt{7}}{4} \cdot \frac{1}{2} \approx 0.99215$$

For the approximation,

$$1 - \frac{\gamma^2}{8km} = 1 - \frac{1/4}{8(1)(4)} = 1 - \frac{1}{128} \approx 0.992187$$

In this case, the approximation is pretty good!

(f) Compare the maximizing value of α with the frequency of the undamped homogeneous DE and the pseudo frequency of the damped system. What should we find?

SOLUTION: The frequency of the undamped system was 2, the frequency of the damped system was slightly less at 1.98, and the frequency α that maximized the amplitude of the particular solution to the damped, forced system was $\alpha = \sqrt{31/8} \approx 1.9685$ These were expected- That is, we did not expect the amplitude of the damped system to be a maximum exactly at 2, but we did expect it to be close. And in fact, it is close to the period of the quasi-periodic solution to the homogeneous equation.