## Solutions to Homework, Section 5.4

We added some and deleted the HW for 5.5, so the homework for this section is:
1, 3, 5, 8, 10, 17, 19, 27, 31, 41

1. $$x^2y'' + 4xy' + 2y = 0$$

   The characteristic equation is:

   $$r(r-1) + 4r + 2 = 0 \quad \Rightarrow \quad r^2 + 3r + 2 = 0 \quad \Rightarrow \quad (r+2)(r+1) = 0$$

   So the general solution is:

   $$y = C_1x^{-1} + C_2x^{-2}$$

3. $$x^2y'' - 3xy' + 4y = 0$$

   As before, go to the characteristic equation:

   $$r(r-1) - 3r + 4 = 0 \quad \Rightarrow \quad r^2 - 4r + 4 = 0 \quad \Rightarrow \quad (r-2)^2 = 0$$

   The solution is:

   $$y = x^2(C_1 + C_2 \ln|x|)$$

5. Similar to 3 (a double root of $r = 1$)

8. The roots to the characteristic equation are

   $$r = \frac{3}{2} \pm \frac{\sqrt{3}}{2}\, i$$

   Therefore, the solutions are found by taking the real part of $x^r$ and the imaginary part of $x^r$. For the sake of completeness, the details are below:

   $$x^r = x^{\frac{3}{2} + \frac{\sqrt{3}}{2}\, i} = x^{3/2} e^{i\, \frac{\sqrt{3}}{2} \ln|x|}$$

   From which we get:

   $$y = x^{3/2}\left(C_1 \cos\left(\frac{\sqrt{3}}{2} \ln|x|\right) + C_2 \sin\left(\frac{\sqrt{3}}{2} \ln|x|\right)\right)$$

10. This one is a bit different. You can let $w = x - 2$, then back-substitute at the end.

    $$w^2y'' + 5wy' + 8y = 0 \quad \Rightarrow \quad r(r-1) + 5r + 8 = 0 \quad \Rightarrow \quad r^2 + 4r + 8 = 0$$

    Complete the square (or use the quadratic formula):

    $$(r^2 + 4r + 4) + 4 = 0 \quad \Rightarrow \quad (r+2)^2 = -4 \quad \Rightarrow \quad r = -2 \pm 2i$$

    Therefore, the solution is:

    $$y = (x-2)^{-2}\left(C_1 \cos(2\ln|x-2|) + C_2 \sin(2\ln|x-2|)\right)$$

17. Find all singular points and determine if they are regular or irregular.

$$xy'' + (1-x)y' + xy = 0$$

The only singular point is $x = 0$. Check the limits:

$$\lim_{x\to 0} x\frac{Q(x)}{P(x)} = \lim_{x\to 0} \frac{x(1-x)}{x} = 1 \qquad \lim_{x\to 0} x^2\frac{R(x)}{P(x)} = \lim_{x\to 0} \frac{x^3}{x} = 0$$

Therefore, $x = 0$ is a regular singular point.

19. Same question as 17:

$$x^2(1-x)y'' + (x-2)y' - 3xy = 0$$

The singular points are $x = 0$ and $x = 1$. First, check $x = 0$:

$$\lim_{x\to 0} x\frac{Q(x)}{P(x)} = \lim_{x\to 0} \frac{x(x-2)}{x^2(1-x)} = \lim_{x\to 0} \frac{(x-2)}{x(1-x)}$$

This fraction goes to "$-2/0$", so that the limit overall does not exist (the limit is $\pm\infty$). Therefore, $x = 0$ is an irregular singular point. Now, at $x = 1$:

$$\lim_{x\to 1}(x-1)\frac{Q(x)}{P(x)} = \lim_{x\to 0} \frac{(x-1)(x-2)}{x^2(1-x)} = 1 \qquad \lim_{x\to 1}(x-1)^2\frac{R(x)}{P(x)} = \lim_{x\to 0} \frac{-3x(x-1)^2}{x^2(1-x)} = 0$$

27. Similar to 19, you should find that both $x = 1$ and $x = -2$ are regular singular points.

31. Hint is that $\sin(x)/x$ goes to 1 as $x \to 0$, so you should find that $x = 0$ is a regular singular point.

41. Substituting the series $\sum a_n x^n$ into the DE will yield the polynomial:

$$3a_1 + \sum_{n=2}^{\infty} \{[2n(n-1) + 3n]\, a_n + a_{n-2}\}\, x^{n-1} = 0$$

Therefore, $a_1 = 0$ and

$$a_n = -\frac{a_{n-2}}{n(2n+1)} \qquad \text{for } n = 2, 3, 4, \cdots$$

We can see that $a_2$ is in terms of $a_0$, $a_3 = 0$, $a_4$ is in terms of $a_0$, $a_5 = 0$, and so on. Therefore, the solution is only in terms of $a_0$ (instead of both $a_0$ and $a_1$, as it is about an ordinary point).