End note for Chapter 5

Given the differential equation with regular singular point $x_0 = 0$:

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

Dividing by $P(x)$ and multiplying by x^2 gives us:

$$y'' + \frac{Q}{P}y' + \frac{R}{P}y = 0 \quad \Rightarrow \quad x^2y'' + x\left(x\frac{Q}{P}\right)y' + \left(x^2\frac{R}{P}\right)y = 0$$

Then we know that the following limits exist:

$$\lim_{x\to 0} x\frac{Q}{P} = p_0 \qquad \lim_{x\to 0} x^2\frac{R}{P} = q_0$$

Consider the Euler Equation associated with the original DE:

$$x^2y'' + p_0xy' + q_0y = 0$$

And the characteristic equation, $r(r-1) + p_0r + q_0 = 0$. The solutions to this equation are the exponents at the singularity and are used to solve the differential equation about a regular singular point. We'll finish here for now.

Worked Example: (Actually Exercise 6, 5.5) Given

$$x^2y'' + xy' + (x-2)y = 0$$

Show that $x = 0$ is a regular singular point, find its associated Euler equation, and find the solutions to the associated Euler equation.

SOLUTION:

$$\lim_{x\to 0} x\frac{Q}{P} = \lim_{x\to 0}\frac{x^2}{x^2} = 1 \qquad \lim_{x\to 0} x^2\frac{R}{P} = \lim_{x\to 0}\frac{x^2(x-2)}{x^2} = -2$$

Therefore, the Euler equation associated with the DE is:

$$x^2y'' + xy' - 2y = 0 \quad \Rightarrow \quad r(r-1) + r - 2 = 0 \quad r = \pm\sqrt{2}$$

And the solution to the associated Euler equation is: $C_1x^{-\sqrt{2}} + C_2x^{\sqrt{2}}$.

Side Remark: If you ask Maple to solve the original differential equation using a series, it gives you a function of the form:

$$y(x) = C_1x^{-\sqrt{2}}\left(\sum_{n=0}^{\infty} a_nx^n\right) + C_2x^{\sqrt{2}}\left(\sum_{n=0}^{\infty} b_nx^n\right)$$

where the coefficients are also given- I just wanted you to see the *form* of the solution- Note the relationship of this full solution to the solution of the associated Euler equation.

Worked Example 2 Same question (and $x = 0$ is the regular singular point):

$$2x^2y'' + 3xy' + (2x^2 - 1)y = 0$$

SOLUTION: Follow the same procedure, and compute the two limits. Those numbers are then used in the Euler Equation:

$$\lim_{x\to 0} x\frac{Q}{P} = \lim_{x\to 0} \frac{3x^2}{2x^2} = \frac{3}{2} \qquad \lim_{x\to 0} x^2\frac{R}{P} = \lim_{x\to 0} \frac{x^2(2x^2-1)}{2x^2} = -\frac{1}{2}$$

Therefore, the associated Euler equation is:

$$x^2y'' + \frac{3}{2}xy' - \frac{1}{2}y = 0 \quad \Rightarrow \quad r(r-1) + \frac{3}{2}r - \frac{1}{2} = 0 \quad \Rightarrow \quad r = \frac{1}{2}, -1$$

The Euler equation has solution $y(x) = C_1\sqrt{x} + C_2x^{-1}$.

Side Remark: Asking Maple to solve the original differential equation yields the series:

$$y(x) = C_1\sqrt{x}\left(\sum_{n=0}^{\infty} a_nx^n\right) + \frac{C_2}{x}\left(\sum_{n=0}^{\infty} b_nx^n\right)$$

(Again, note the relationship to the associated Euler equation).

Exercises

For each differential equation below, show that $x = 0$ is a regular singular point, then find the solution to the associated Euler equation.

1. $3x^2y'' + 2xy' + x^2y = 0$
2. $xy'' + (1-x)y' - y = 0$
3. $x^2y'' - x(x+3)y' + (x+3)y = 0$

SOLUTIONS:

(Only the characteristic equation and solutions are given)

1. $r(r-1) + \frac{2}{3}r + 0 = 0$, so $y(x) = C_1x^{1/3} + C_2$
2. $r(r-1) + r + 0 = 0$, so $y(x) = C_1 + C_2\ln|x|$ (It's OK if you assume $x > 0$).
3. $r(r-1) - 3r + 3 = 0$, so $y(x) = C_1x^3 + C_2x$