Sample Questions (Chapter 3, Math 244)

1. State the Existence and Uniqueness theorem for linear, second order differential equations (non-homogeneous is the most general form):

2. True or False?

   (a) The characteristic equation for $y'' + y' + y = 1$ is $r^2 + r + 1 = 1$

   (b) The characteristic equation for $y'' + xy' + e^x y = 0$ is $r^2 + xr + e^x = 0$

   (c) The function $y = 0$ is always a solution to a second order linear homogeneous differential equation.

   (d) In using the Method of Undetermined Coefficients, the ansatz $y_p = (Ax^2 + Bx + C)(D\sin(x) + E\cos(x))$ is equivalent to

   $$y_p = (Ax^2 + Bx + C)\sin(x) + (Dx^2 + Ex + F)\cos(x)$$

3. Find values of a for which **any** solution to:

   $$y'' + 10y' + ay = 0$$

   will tend to zero (that is, $\lim_{t\to 0} y(t) = 0$.

4.
   - Compute the Wronskian between $f(x) = \cos(x)$ and $g(x) = 1$.
   - Can these be two solutions to a second order linear homogeneous differential equation? Be specific. (Hint: Abel's Theorem)

5. Construct the operator associated with the differential equation: $y' = y^2 - 4$. Is the operator linear? Show that your answer is true by using the definition of a linear operator.

6. Find the solution to the initial value problem:

   $$u'' + u = \begin{cases} 3t & \text{if } 0 \le t \le \pi \\ 3(2\pi - t) & \text{if } \pi < t < 2\pi \\ 0 & \text{if } t \ge 2\pi \end{cases} \qquad u(0) = 0 \quad u'(0) = 0$$

7. Solve: $u'' + \omega_0^2 u = F_0 \cos(\omega t)$, $\quad u(0) = 0 \quad u'(0) = 0$ if $\omega \neq \omega_0$ using the Method of Undetermined Coefficients.

8. Compute the solution to: $u'' + \omega_0^2 u = F_0 \cos(\omega_0 t) \qquad u(0) = 0 \quad u'(0) = 0$ two ways:
   - Start over, with Method of Undetermined Coefficients
   - Take the limit of your answer from Question 6 as $\omega \to \omega_0$.

9. For the following question, recall that the acceleration due to gravity is 32 ft/sec^2.

   An 8 pound weight is attached to a spring from the ceiling. When the weight comes to rest at equilibrium, the spring has been stretched 2 feet. The damping constant for the system is 1−lb-sec/ft. If the weight is raised 6 inches above equilibrium and given an upward velocity of 1 ft/sec, find the equation of motion for the weight. Write the solution as $R\cos(\omega t - \delta)$, if possible.

10. Given that $y_1 = \frac{1}{t}$ solves the differential equation:

$$t^2 y'' - 2y = 0$$

   Find a fundamental set of solutions.

11. Suppose a mass of 0.01 kg is suspended from a spring, and the damping factor is $\gamma = 0.05$. If there is no external forcing, then what would the spring constant have to be in order for the system to *critically damped? underdamped?*

12. Give the full solution, using any method(s). If there is an initial condition, solve the initial value problem.

   (a) $y'' + 4y' + 4y = t^{-2}\mathrm{e}^{-2t}$

   (b) $y'' - 2y' + y = t\mathrm{e}^t + 4$, $y(0) = 1$, $y'(0) = 1$.

   (c) $y'' + 4y = 3\sin(2t)$, $y(0) = 2$, $y'(0) = -1$.

   (d) $y'' + 9y = \sum_{m=1}^{N} b_m \cos(m\pi t)$

13. Rewrite the expression in the form $a + ib$: (i) 2^{i-1} (ii) $\mathrm{e}^{(3-2i)t}$ (iii) $\mathrm{e}^{i\pi}$

14. Write $a + ib$ in polar form: (i) $-1 - \sqrt{3}i$ (ii) $3i$ (iii) -4 (iv) $\sqrt{3} - i$

15. Find a second order linear differential equation with constant coefficients whose general solution is given by:

$$y(t) = C_1 + C_2\mathrm{e}^{-t} + \frac{1}{2}t^2 - t$$

16. Determine the longest interval for which the IVP is certain to have a unique solution (Do not solve the IVP):

$$t(t-4)y'' + 3ty' + 4y = 2 \qquad y(3) = 0 \quad y'(3) = -1$$

17. Let $L(y) = ay'' + by' + cy$ for some value(s) of a, b, c.

   If $L(3\mathrm{e}^{2t}) = -9\mathrm{e}^{2t}$ and $L(t^2 + 3t) = 5t^2 + 3t - 16$, what is the particular solution to:

$$L(y) = -10t^2 - 6t + 32 + \mathrm{e}^{2t}$$

18. If we take the ansatz $y = t^r$, find a fundamental set of solutions for

$$t^2 y'' + 2ty' - 2y = 0$$

19. If $x = \ln(t)$ and $\dot{y} = dy/dt$, then verify that

$$\frac{dy}{dx} = t\,\dot{y} \quad \text{and} \quad \frac{d^2y}{dx^2} = t^2\ddot{y} + t\dot{y}$$

20. Use Variation of Parameters to find a particular solution to the following, then verify your answer using the Method of Undetermined Coefficients:

$$4y'' - 4y' + y = 16e^{t/2}$$

21. Compute the Wronskian of two solutions of the given DE without solving it:

$$x^2 y'' + xy' + (x^2 - \alpha^2)y = 0$$

22. If $y'' - y' - 6y = 0$, with $y(0) = 1$ and $y'(0) = \alpha$, determine the value(s) of α so that the solution tends to zero as $t \to \infty$.

23. Show that the period of motion of an undamped vibration of a mass hanging from a vertical spring is $2\pi\sqrt{L/g}$, where L is the elongation of the spring due to the mass and g is the acceleration due to gravity.

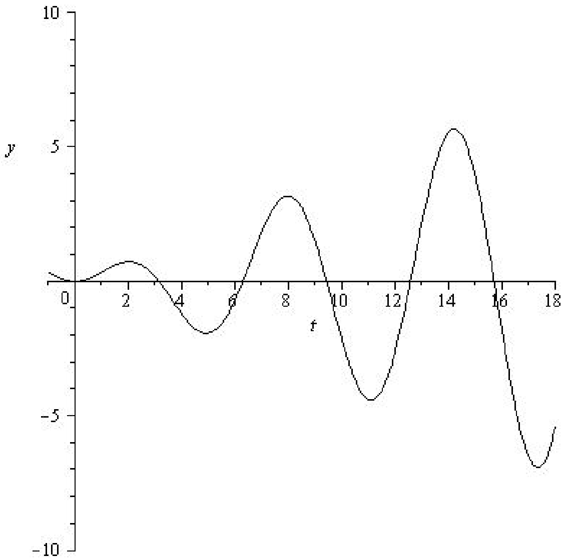
24. Give the general solution to $y'' + y = \frac{1}{\sin(t)} + t$

25. A mass of 0.5 kg stretches a spring to 0.05 meters. (i) Find the spring constant. (ii) Does a stiff spring have a large spring constant or a small spring constant (explain).

26. A mass of $\frac{1}{2}$ kg is attached to a spring with spring constant 2 (kg/sec^2). The spring is pulled down an additional 1 meter then released. Find the equation of motion if the damping constant is $c = 2$ as well:

27. Match the solution curve to its IVP (There is one DE with no graph, and one graph with no DE- You should not try to completely solve each DE).

    (a) $5y'' + y' + 5y = 0$, $y(0) = 10$, $y'(0) = 0$
    (b) $y'' + 5y' + y = 0$, $y(0) = 10$, $y'(0) = 0$
    (c) $y'' + y' + \frac{5}{4}y = 0$, $y(0) = 10$, $y'(0) = 0$
    (d) $5y'' + 5y = 4\cos(t)$, $y(0) = 0$, $y'(0) = 0$
    (e) $y'' + \frac{1}{2}y' + 2y = 10$, $y(0) = 0$, $y'(0) = 0$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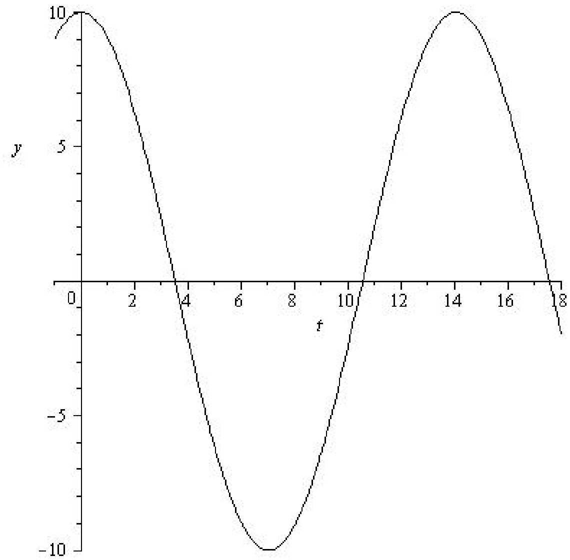
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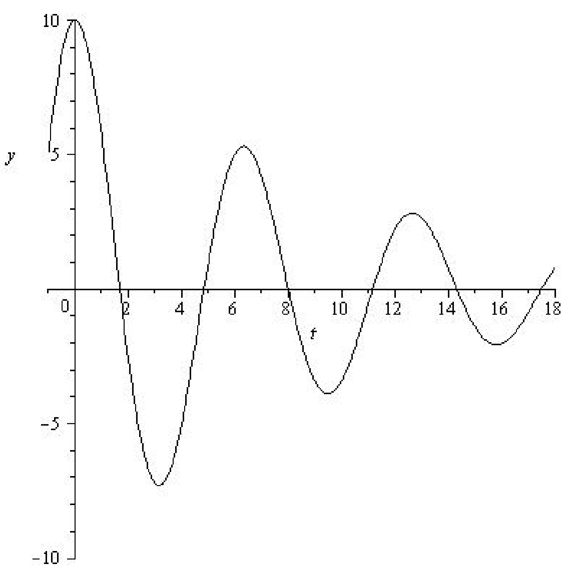
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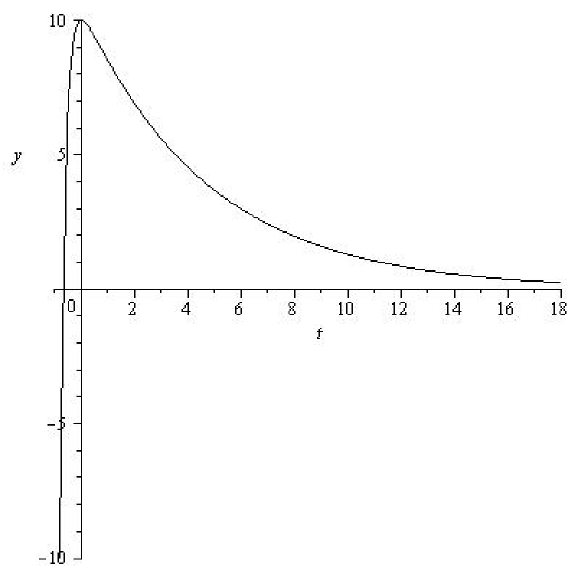
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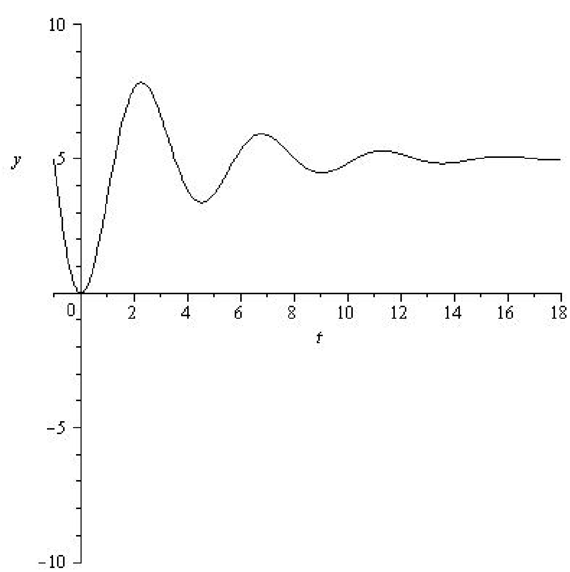
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