Quiz 5 Solutions

1. Solve the following IVP using the Method of Undetermined Coefficients:

$$y'' + 4y = t^2 + 3e^t \qquad y(0) = 0 \quad y'(0) = 2$$

SOLUTION: First get the homogeneous part, then the particular part of the solution. Then solve using the initial conditions (ICs).

- $y_h(t) = C_1\cos(2t) + C_2\sin(2t)$
- For the particular solution, use the Method of undetermined coefficients:
  - For t^2, use the ansatz $y_{p_1} = At^2 + Bt + C$. Then:

$$\begin{array}{rcllc} 4y_p & = & 4At^2+ & 4Bt+ & 4C \\ y_p'' & = & & & 2A \\ \hline t^2 & = & 4At^2+ & 4Bt+ & (4C+2A) \end{array} \quad \Rightarrow \quad A = \frac{1}{4}, \quad B = 0, \quad C = -\frac{1}{8}$$

  - For $3e^t$, guess $y_{p_2} = Ae^t$ so that

$$y'' + 4y = Ae^t + 4Ae^t = 3e^t \quad \Rightarrow \quad A = \frac{3}{5}$$

  So, the particular solution is $\frac{1}{4}t^2 - \frac{1}{8} + \frac{3}{5}e^t$.
- Solve for the constants:

$$y = C_1\cos(2t) + C_2\sin(2t) + \frac{1}{4}t^2 - \frac{1}{8} + \frac{3}{5}e^t \quad \Rightarrow \quad 0 = C_1 - \frac{1}{8} + \frac{3}{5} \Rightarrow C_1 = -\frac{19}{40}$$

  And, for the derivative:

$$y' = -2C_1\sin(2t) + 2C_2\cos(2t) + \frac{1}{2}t + \frac{3}{5}e^t \quad \Rightarrow \quad 2 = 2C_2 + \frac{3}{5} \quad \Rightarrow \quad C_2 = \frac{7}{10}$$

2. Give the general solution to the following using Variation of Parameters. Assume the functions y_1, y_2 are solutions to the homogeneous equation (you can verify this):

$$x^2y'' - 3xy' + 4y = x^2\ln(x), \quad x > 0, \quad \text{with} \quad y_1 = x^2, \qquad y_2 = x^2\ln(x)$$

SOLUTION: First, put the equation in standard form, then apply the formulas:

$$y'' - \frac{3}{x}y' + \frac{4}{x^2}y = \ln(x)$$

with y_1, y_2 given and $g = \ln(x)$ and

$$W(y_1, y_2) = \begin{vmatrix} x^2 & x^2\ln(x) \\ 2x & 2x\ln(x) + x \end{vmatrix} = x^3$$

Now,

$$u_1' = -\frac{x^2\ln(x)\cdot\ln(x)}{x^3} = -\frac{1}{x}(\ln(x))^2 \quad \Rightarrow \quad u_1 = -\frac{1}{3}(\ln(x))^3$$

$$u_2' = \frac{x^2\cdot\ln(x)}{x^3} \quad \Rightarrow \quad u_2 = \frac{1}{2}(\ln(x))^2$$

Therefore

$$y_p = -\frac{1}{3}(\ln(x))^3\cdot x^2 + \frac{1}{2}(\ln(x))^2\cdot x^2\ln(x) = \frac{1}{6}x^2(\ln(x))^3$$

Don't forget to give the full general solution now:

$$y(t) = C_1x^2 + C_2x^2\ln(x) + \frac{1}{6}x^2(\ln(x))^3$$

3. Let $y'' + 9y = F(t)$ with $y(0) = 0$ and $y'(0) = 0$.

   (a) Give an example $F(t)$ for which the solution is periodic with period $2\pi/3$.

   SOLUTION: The period of the homogeneous part of the solution is $2\pi/3$. If we used that for the forcing function, we would get resonance (which would NOT be periodic). However, from the method of undetermined coefficients we know that, for example, if $F(t) = 1$, then the solution would be the homogeneous solution translated upward, which still has the same period as before.

   Of course, we didn't rule out a trivial forcing ($F = 0$) either.

   (b) Show that no function of the form $F(t) = A\cos(\alpha t)$ can be found so that the period of the solution $y(t)$ is exactly 3π.

   SOLUTION: For the period of the solution to be 3π, we would have to be able to find positive integers k, n so that the following would be true:

$$\frac{2\pi}{\alpha}k = \frac{2\pi}{3}n = 3\pi$$

   But this last equality implies that $n = \frac{9}{2}$, which is not an integer. Therefore, we would not be able to find α so that the period of the solution is 3π.

   (c) Give an example of a function $F(t)$ so that $|F(t)| < 2$ for all t, and y is bounded and periodic.

   SOLUTION: Just about any bounded function F will do- Just avoid anything that is periodic with period $2\pi/3$ (see next question).

   (d) Similarly, give an example of a function $F(t)$ so that $|F(t)| < 2$ for all t, but y tends to infinity as $t \to \infty$.

   SOLUTION: Any forcing with period $2\pi/3$ will do, such as:

$$y'' + 9y = \sin(3t) + \cos(3t)$$

   In this case, the particular part of the solution is:

$$t\left(\frac{1}{6}\sin(3t) - 3\cos(3t)\right)$$

4. In Section 3.7, p. 198, the author states that

$$\left(1 - \frac{\gamma^2}{4km}\right)^{1/2} \approx 1 - \frac{\gamma^2}{8km}$$

   when $\gamma^2/4km$ is small. Prove it (Hint: Think tangent line approximation).

   SOLUTION: Consider the function $\sqrt{1-x}$ at $x = 0$ ($y = 1$). The slope of the tangent line at $(0, 1)$ is:

$$\frac{1}{2}(1-x)^{-1/2}(-1)\bigg|_{x=0} = -\frac{1}{2}$$

   Therefore, the tangent line is: $y - 1 = -\frac{1}{2}(x - 0)$, or $y = 1 - \frac{1}{2}x$, which is the approximation given: That is, we have shown that for x small,

$$\sqrt{1-x} \approx 1 - \frac{1}{2}x$$

   Now, if we let

$$x = \frac{\gamma^2}{4km}$$

   then we get our textbook authors' claim.