Sample Questions (Chapter 3, Math 244)

Exam Notes: You will not be allowed to have a calculator or any notes. You will be provided with the system of equations for Variation of Parameters. Be sure to go through your old quizzes and homework.

1. State the Existence and Uniqueness theorem for linear, second order differential equations (non-homogeneous is the most general form):

2. True or False?

   (a) The characteristic equation for $y'' + y' + y = 1$ is $r^2 + r + 1 = 1$

   (b) The characteristic equation for $y'' + xy' + e^x y = 0$ is $r^2 + xr + e^x = 0$

   (c) The function $y = 0$ is always a solution to a second order linear homogeneous differential equation.

   (d) In using the Method of Undetermined Coefficients, the ansatz $y_p = (Ax^2 + Bx + C)(D\sin(x) + E\cos(x))$ is equivalent to

   $$y_p = (Ax^2 + Bx + C)\sin(x) + (Dx^2 + Ex + F)\cos(x)$$

   (e) Consider the function:

   $$y(t) = \cos(t) - \sin(t)$$

   Then amplitude is 1, the period is 1 and the phase shift is 0.

   (f) If $y'' + y' + 9y = cos(\omega t)$, we have resonance if $\omega = 3$.

3. Find values of a for which **any** solution to:

   $$y'' + 10y' + ay = 0$$

   will tend to zero (that is, $\lim_{t\to 0} y(t) = 0$.

4.
   - Compute the Wronskian between $f(x) = \cos(x)$ and $g(x) = 1$.
   - Can these be two solutions to a second order linear homogeneous differential equation? Be specific. (Hint: Abel's Theorem)

5. Construct the operator associated with the differential equation: $y' = y^2 - 4$. Is the operator linear? Show that your answer is true by using the definition of a linear operator.

6. Find the solution to the initial value problem:

$$u'' + u = \begin{cases} 3t & \text{if } 0 \le t \le \pi \\ 3(2\pi - t) & \text{if } \pi < t < 2\pi \\ 0 & \text{if } t \ge 2\pi \end{cases} \qquad u(0) = 0 \quad u'(0) = 0$$

7. Solve: $u'' + \omega_0^2 u = F_0 \cos(\omega t), \quad u(0) = 0 \quad u'(0) = 0$ if $\omega \neq \omega_0$ using the Method of Undetermined Coefficients.

8. Compute the solution to: $u'' + \omega_0^2 u = F_0 \cos(\omega_0 t) \qquad u(0) = 0 \quad u'(0) = 0$ two ways:

   - Start over, with Method of Undetermined Coefficients
   - Take the limit of your answer from Question 6 as $\omega \to \omega_0$.

9. For the following question, recall that the acceleration due to gravity is 32 ft/sec^2.

   An 8 pound weight is attached to a spring from the ceiling. When the weight comes to rest at equilibrium, the spring has been stretched 2 feet. The damping constant for the system is 1−lb-sec/ft. If the weight is raised 6 inches above equilibrium and given an upward velocity of 1 ft/sec, find the equation of motion for the weight.

10. Given that $y_1 = \frac{1}{t}$ solves the differential equation:

$$t^2 y'' - 2y = 0$$

    Find a fundamental set of solutions using Abel's Theorem.

11. Suppose a mass of 0.01 kg is suspended from a spring, and the damping factor is $\gamma = 0.05$. If there is no external forcing, then what would the spring constant have to be in order for the system to *critically damped? underdamped?*

12. Give the full solution, using any method(s). If there is an initial condition, solve the initial value problem.

    (a) $y'' + 4y' + 4y = t^{-2} e^{-2t}$

    (b) $y'' - 2y' + y = te^t + 4$, $y(0) = 1$, $y'(0) = 1$.

    (c) $y'' + 4y = 3\sin(2t)$, $y(0) = 2$, $y'(0) = -1$.

    (d) $y'' + 9y = \sum_{m=1}^{N} b_m \cos(m\pi t)$

13. Rewrite the expression in the form $a + ib$: (i) 2^{i-1} (ii) $e^{(3-2i)t}$ (iii) $e^{i\pi}$

14. Write $a + ib$ in polar form: (i) $-1 - \sqrt{3}i$ (ii) $3i$ (iii) -4 (iv) $\sqrt{3} - i$

15. Find a second order linear differential equation with constant coefficients whose general solution is given by:

$$y(t) = C_1 + C_2 e^{-t} + \frac{1}{2}t^2 - t$$

16. Determine the longest interval for which the IVP is certain to have a unique solution (Do not solve the IVP):

$$t(t-4)y'' + 3ty' + 4y = 2 \qquad y(3) = 0 \quad y'(3) = -1$$

17. Let $L(y) = ay'' + by' + cy$ for some value(s) of a, b, c.
    If $L(3e^{2t}) = -9e^{2t}$ and $L(t^2 + 3t) = 5t^2 + 3t - 16$, what is the particular solution to:

$$L(y) = -10t^2 - 6t + 32 + e^{2t}$$

18. Solve the following Euler equations:
    (a) $t^2y'' + 2ty' - 2y = 0$
    (b) $t^2y'' + ty' + 9y = 0$

19. Consider the differential equation:

$$u'' + u' + 2u = \cos(\omega t)$$

    (a) Solve the homogeneous equation.
    (b) It can be shown that the amplitude of the steady state (or particular part of the solution) is given by:

$$R = \frac{1}{\sqrt{\omega^4 - 3\omega^2 + 4}}$$

    Find the critical point for R (in terms of ω).
    (c) If we were to change the damping factor so that it becomes closer and closer to zero, what would we expect would happen to our answer to the previous question and to the amplitude R?

20. Explain in words the concepts of beating and resonance, and how they relate to each other.

21. Use Variation of Parameters to find a particular solution to the following, then verify your answer using the Method of Undetermined Coefficients:

$$4y'' - 4y' + y = 16e^{t/2}$$

22. Compute the Wronskian of two solutions of the given DE without solving it:

$$x^2y'' + xy' + (x^2 - \alpha^2)y = 0$$

23. If $y'' - y' - 6y = 0$, with $y(0) = 1$ and $y'(0) = \alpha$, determine the value(s) of α so that the solution tends to zero as $t \to \infty$.

24. Give the general solution to $y'' + y = \frac{1}{\sin(t)} + t$

25. A mass of 0.5 kg stretches a spring to 0.05 meters. (i) Find the spring constant. (ii) Does a stiff spring have a large spring constant or a small spring constant (explain).

26. A mass of 5 kg stretches a spring 0.1 m. The mass is acted on by an external force of $10\sin(t/2)$ N (newtons) and moves in a medium that imparts a viscous force of 2 N when the speed of the mass is 0.04 meters per second. If the mass is set in motion by pulling down 0.3 meters and imparting a velocity of 0.03 meters per second, formulate the initial value problem describing the motion of the mass (do not solve).

27. Match the solution curve to its IVP (There is one DE with no graph, and one graph with no DE- You should not try to completely solve each DE).

   (a) $5y'' + y' + 5y = 0,\ y(0) = 10,\ y'(0) = 0$
   (b) $y'' + 5y' + y = 0,\ y(0) = 10,\ y'(0) = 0$
   (c) $y'' + y' + \frac{5}{4}y = 0,\ y(0) = 10,\ y'(0) = 0$
   (d) $5y'' + 5y = 4\cos(t),\ y(0) = 0,\ y'(0) = 0$
   (e) $y'' + \frac{1}{2}y' + 2y = 10,\ y(0) = 0,\ y'(0) = 0$