Sample Question Solutions (Chapter 3, Math 244)

1. State the Existence and Uniqueness theorem for linear, second order differential equations (non-homogeneous is the most general form):

   SOLUTION:

   Let $y'' + p(t)y' + q(t)y = g(t)$, with $y(t_0) = y_0$ and $y'(t_0) = v_0$. Then if p, q and g are all continuous on an open interval I containing t_0, a unique solution exists to the IVP, valid for all t in I.

2. True or False?

   (a) The characteristic equation for $y'' + y' + y = 1$ is $r^2 + r + 1 = 1$
   SOLUTION: False. The characteristic equation is for the homogeneous equation, $r^2 + r + 1 = 0$

   (b) The characteristic equation for $y'' + xy' + e^x y = 0$ is $r^2 + xr + e^x = 0$
   SOLUTION: False. The characteristic equation was defined only for DEs with constant coefficients.

   (c) The function $y = 0$ is always a solution to a second order linear homogeneous differential equation.
   SOLUTION: True. It is true generally- If L is a linear operator, then $L(0) = 0$.

   (d) In using the Method of Undetermined Coefficients, the ansatz $y_p = (Ax^2 + Bx + C)(D\sin(x) + E\cos(x))$ is equivalent to

   $$y_p = (Ax^2 + Bx + C)\sin(x) + (Dx^2 + Ex + F)\cos(x)$$

   SOLUTION: False- We have to be able to choose the coefficients for each polynomial (for the sine and cosine) independently of each other. In the form:

   $$(Ax^2 + Bx + C)(D\sin(x) + E\cos(x))$$

   the polynomials for the sine and cosine are constant multiples of each other, which may not necessarily hold true. That's why we need one polynomial for the sine, and one for the cosine (so the second guess is the one to use).

   (e) Consider the function:

   $$y(t) = \cos(t) - \sin(t)$$

   Then amplitude is 1, the period is 1 and the phase shift is 0.
   SOLUTION: False. For this question to make sense, we have to first write the function as $R\cos(\omega(t - \delta))$. In this case, the amplitude is R:

   $$R = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

   The period is 2π (the circular frequency, or natural frequency, is 1), and the phase shift δ is:

   $$\tan(\delta) = -1 \quad \Rightarrow \quad \delta = -\frac{\pi}{4}$$

(f) If $y'' + y' + 9y = \cos(\omega t)$, we have resonance if $\omega = 3$.
SOLUTION: False. With the addition of damping (in this case, $\gamma = 1$), there is not resonance in the usual sense. However, the amplitude of the particular part of the solution will depend on ω, and the value of ω that gives the maximum amplitude is close to 3 (but not exactly 3). Generally speaking, the value of ω giving the maximum amplitude to the particular part of the solution will be something close to (but not exactly) $\sqrt{k/m}$.

3. Find values of a for which **any** solution to:

$$y'' + 10y' + ay = 0$$

will tend to zero (that is, $\lim_{t\to\infty} y(t) = 0$.

SOLUTION: Use the characteristic equation and check the 3 cases (for the discriminant). That is,

$$r^2 + 10r + a = 0 \quad \Rightarrow \quad r = \frac{-10 \pm \sqrt{100 - 4a}}{2}$$

We check some special cases:

- If $100 - 4a = 0$ (or $a = 25$), we get a double root, $r = -5, -5$, or $y_h = \mathrm{e}^{-5t}(C_1 + C_2 t)$, and all solutions tend to zero.
- If the roots are complex, then we can write $r = -5 \pm \beta i$, and we get

$$y_h = \mathrm{e}^{-5t}(C_1 \cos(\beta t) + C_2 \sin(\beta t))$$

  and again, this will tend to zero for any choice of C_1, C_2.
- In the case that $a < 25$, we have to be a bit careful. While it is true that both roots will be *real*, we also want them to both be *negative* for all solutions to tend to zero.
  - When will they both be negative? If $100 - 4a < 100$ (or $\sqrt{100 - 4a} < 10$). This happens as long as $a > 0$.
  - If $a = 0$, the roots will be $r = -10, 0$, and $y_h = C_1\mathrm{e}^{-10t} + C_2$- Therefore, I could choose $C_1 = 0$ and $C_2 \neq 0$, and my solution will not go to zero.
  - If $a < 0$, the roots will be mixed in sign (one positive, one negative), so the solutions will not all tend to zero.

CONCLUSION: If $a > 0$, all solutions to the homogeneous will tend to zero.

4. - Compute the Wronskian between $f(x) = \cos(x)$ and $g(x) = 1$.
   SOLUTION: $W(\cos(x), 1) = \sin(x)$

- Can these be two solutions to a second order linear homogeneous differential equation? Be specific. (Hint: Abel's Theorem)

  SOLUTION: Abel's Theorem tells us that the Wronskian between two solutions to a second order linear homogeneous DE will either be identically zero or never zero on the interval on which the solution(s) are defined.

  Therefore, as long as the interval for the solutions do not contain a multiple of π (for example, $(0, \pi)$, $(\pi, 2\pi)$, etc), then it is possible for the Wronskian for two solutions to be $\sin(x)$.

5. Construct the operator associated with the differential equation: $y' = y^2 - 4$. Is the operator linear? Show that your answer is true by using the definition of a linear operator.

   SOLUTION: The operator is found by getting all terms in y to one side of the equation, everything else on the other. In this case, we have:

   $$L(y) = y' - y^2$$

   This is not a linear operator. We can check using the definition:

   $$L(cy) = cy' - c^2y^2 \neq cL(y)$$

   Furthermore,

   $$L(y_1 + y_2) = (y_1' + y_2') - (y_1 + y_2)^2 \neq L(y_1) + L(y_2)$$

6. Find the solution to the initial value problem:

   $$u'' + u = \begin{cases} 3t & \text{if } 0 \leq t \leq \pi \\ 3(2\pi - t) & \text{if } \pi < t < 2\pi \\ 0 & \text{if } t \geq 2\pi \end{cases} \qquad u(0) = 0 \quad u'(0) = 0$$

   SOLUTION: Without regards to the initial conditions, we can solve the three non-homogeneous equations. In each case, the homogeneous part of the solution is $c_1 \cos(t) + c_2 \sin(t)$.

   - $u'' + u = 3t$. We would start with $y_p = At + B$. Substituting, we get: $At + B = 3t$, so $A = 3$ and $B = 0$- Therefore, the general solution in this case is:

     $$u(t) = c_1 \cos(t) + c_2 \sin(t) + 3t$$

   - $u'' + u = 6\pi - 3t$. From our previous analysis, the solution is:

     $$u(t) = c_1 \cos(t) + c_2 \sin(t) + 6\pi - 3t$$

- The last part is just the homogeneous equation.

The only thing left is to find c_1, c_2 in each of the three cases so that the overall function u is continuous:

- $u(0) = 0$, $u'(0) = 0 \Rightarrow$

$$u(t) = -3\sin(t) + 3t \qquad 0 \le t \le \pi$$

- $u(\pi) = 3\pi$ and $u'(\pi) = 6$, so:

$$u(t) = -9\sin(t) + 6\pi - 3t \qquad \pi < t < 2\pi$$

- $u(2\pi) = 0$, $u'(2\pi) = -12$:

$$u(t) = -12\sin(t) \qquad t \ge 2\pi$$

7. Solve: $u'' + \omega_0^2 u = F_0 \cos(\omega t)$, $\quad u(0) = 0 \quad u'(0) = 0$ if $\omega \neq \omega_0$ using the Method of Undetermined Coefficients.

   SOLUTION: The characteristic equation is: $r^2 + \omega_0^2 = 0$, or $r = \pm\omega_0 i$. Therefore,

$$u_h = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t)$$

   Using the Method of Undetermined Coefficients, $u_p = A\cos(\omega t) + B\sin(\omega t)$, and we put that into the DE:

$$\begin{array}{rlr} \omega_0^2 u & = A\omega_0^2 \cos(\omega t) & +B\omega_0^2 \sin(\omega t) \\ u'' & = -A\omega^2 \cos(\omega t) & -B\omega^2 \sin(\omega t) \\ \hline F_0 \cos(\omega t) & = A(\omega_0^2 - \omega^2)\cos(\omega t) & +B(\omega_0^2 - \omega^2)\sin(\omega t) \end{array}$$

   Therefore,

$$A = \frac{F_0}{\omega_0^2 - \omega^2} \qquad B = 0$$

   so that the overall solution is:

$$u(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) + \frac{F_0}{\omega_0^2 - \omega^2}\cos(\omega t)$$

   Put in the initial conditions $u(0) = 0$ and $u'(0) = 0$ to see that $C_1 = -\frac{F_0}{\omega_0^2 - \omega^2}$ and $C_2 = 0$.

8. Compute the solution to: $u'' + \omega_0^2 u = F_0 \cos(\omega_0 t) \qquad u(0) = 0 \quad u'(0) = 0$ two ways:

- Start over, with Method of Undetermined Coefficients

  SOLUTION: The part that changes is the particular part of the solution- We have to multiply by t: Let $u_p = At\cos(\omega_0 t) + Bt\sin(\omega_0 t)$. Then:

$$\begin{array}{rcll}
\omega_0^2 u_p & = & (A\omega_0^2 t \qquad\qquad)\cos(\omega_0 t) & +(B\omega_0^2 t \qquad\qquad)\sin(\omega_0 t) \\
u_p'' & = & (-A\omega_0^2 t + 2B\omega_0)\cos(\omega_0 t) & +(-B\omega_0^2 t - 2Aw)\sin(\omega_0 t) \\
\hline
F_0\cos(\omega_0 t) & = & 2B\omega_0\cos(\omega_0 t) & -2A\omega_0\sin(\omega_0 t)
\end{array}$$

  Therefore, $B = \frac{F_0}{2\omega_0}$ and $A = 0$, so that

$$u(t) = C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t) + \frac{F_0}{2\omega_0} t\sin(\omega_0 t)$$

  Taking into account the initial conditions, we get $C_1 = C_2 = 0$.

- Take the limit of your answer from Question 6 as $\omega \to \omega_0$ (Typo: Should be Question 7).

  SOLUTION:

$$\lim_{\omega\to\omega_0} \frac{F_0(\cos(\omega t) - \cos(\omega_0 t))}{\omega_0^2 - \omega^2} =?$$

  We can use l'Hospital's Rule (differentiate with respect to ω!):

$$= \lim_{\omega\to\omega_0} \frac{-F_0 t\sin(\omega t)}{-2\omega} = \frac{F_0}{2\omega_0 t}\sin(\omega_0 t)$$

9. For the following question, recall that the acceleration due to gravity is 32 ft/sec^2.

   An 8 pound weight is attached to a spring from the ceiling. When the weight comes to rest at equilibrium, the spring has been stretched 2 feet. The damping constant for the system is 1−lb-sec/ft. If the weight is raised 6 inches above equilibrium and given an upward velocity of 1 ft/sec, find the equation of motion for the weight. Write the solution as $R\cos(\omega t - \delta)$, if possible.

   SOLUTION: First the constants. Since $mg - kL = 0$, we find that $mg = 8$, so $2k = 8$, or $k = 4$.

   We are given that $\gamma = 1$, and since $8 = mg$, then $m = 8/32 = 1/4$.

$$\frac{1}{4}y'' + y' + 4y = 0 \qquad y(0) = -\frac{1}{2} \quad y'(0) = -1$$

   Or we could write: $y'' + 4y' + 16y = 0$. Solving the characteristic equation, we get

$$r^2 + 4r + 16 = 0 \quad \Rightarrow \quad r = \frac{-4 \pm \sqrt{4^2 - 4\cdot 16}}{2} = \frac{-4 \pm 4\sqrt{3}\,i}{2} = -2 \pm 2\sqrt{3}\,i$$

   Therefore, the general solution is:

$$y(t) = e^{-2t}\left(C_1\cos(2\sqrt{3}t) + C_2\sin(2\sqrt{3}\,t)\right)$$

Solving the IVP, differentiate to get the equations for C_1, C_2:

$$y' = -2e^{-2t}\left(C_1\cos(2\sqrt{3}t) + C_2\sin(2\sqrt{3}\,t\right) + 2\sqrt{3}e^{-2t}\left(C_2\cos(2\sqrt{3}t) - C_1\sin(2\sqrt{3}\,t\right)$$

Therefore,

$$\begin{aligned} -\tfrac{1}{2} &= C_1 \\ -1 &= -2C_1 + 2\sqrt{3}C_2 \end{aligned} \quad \Rightarrow \quad C_1 = -\frac{1}{2} \quad C_2 = -\frac{1}{\sqrt{3}}$$

10. Given that $y_1 = \frac{1}{t}$ solves the differential equation:

$$t^2 y'' - 2y = 0$$

Find a fundamental set of solutions.

SOLUTION: I like using the Wronskian for these-

First, rewrite the differential equation in standard form:

$$y'' - \frac{2}{t^2}y = 0$$

Then $p(t) = 0$ and $W(y_1, y_2) = Ce^0 = C$. On the other hand, the Wronskian is:

$$W(y_1, y_2) = \frac{1}{t}y_2' + \frac{1}{t^2}y_2$$

Put these together:

$$\frac{1}{t}y_2' + \frac{1}{t^2}y_2 = C \quad y_2' + \frac{1}{t}y_2 = Ct$$

The integrating factor is t,

$$(ty_2)' = Ct^2 \quad \Rightarrow \quad ty_2 = C_1t^3 + C_2 \quad \Rightarrow \quad C_1t^2 + \frac{C_2}{t}$$

Notice that we have *both* parts of the homogeneous solution, $y_1 = \frac{1}{t}$ and $y_2 = t^2$.

Alternative Solution: Use Reduction of Order, where we assume that

$$y_2 = vy_1 = \frac{v}{t} \quad \Rightarrow \quad y_2' = \frac{v't - v}{t^2} = \frac{v'}{t} - \frac{v}{t^2} \quad \Rightarrow \quad y_2'' = \frac{v''}{t} - 2\frac{v'}{t^2} + 2\frac{v}{t^3}$$

Substituting this back into the DE, we get

$$tv'' - 2v' = 0 \quad \Rightarrow \quad \frac{v'}{t^2} = C_1 \quad \Rightarrow \quad v = \frac{C_1}{3}t^3 + C_2 \quad \Rightarrow \quad y_2 = C_3t^2 + \frac{C_2}{t}$$

and again we see that we can take $y_2 = t^2$.

11. Suppose a mass of 0.01 kg is suspended from a spring, and the damping factor is $\gamma = 0.05$. If there is no external forcing, then what would the spring constant have to be in order for the system to *critically damped? underdamped?*

SOLUTION: We can find the differential equation:

$$0.01u'' + 0.05u' + ku = 0 \quad \Rightarrow \quad u'' + 5u' + 100ku = 0$$

Then the system is *critically damped* if the discriminant (from the quadratic formula) is zero:

$$b^2 - 4ac = 25 - 4 \cdot 100k = 0 \quad \Rightarrow \quad k = \frac{25}{400} = \frac{1}{16}$$

The system is *underdamped* if the discriminant is negative:

$$25 - 400k < 0 \quad \Rightarrow \quad k > \frac{1}{16}$$

12. Give the full solution, using any method(s). If there is an initial condition, solve the initial value problem.

(a) $y'' + 4y' + 4y = t^{-2}\mathrm{e}^{-2t}$

Using the Variation of Parameters, $y_p = u_1y_1 + u_2y_2$, we have:

$$y_1 = \mathrm{e}^{-2t} \quad y_2 = t\mathrm{e}^{-2t} \quad g(t) = \frac{\mathrm{e}^{-2t}}{t^2}$$

with a Wronskian of e^{-4t}. You should find that:

$$u_1' = -\frac{1}{t} \qquad u_2' = \frac{1}{t^2}$$

$$u_1 = -\ln(t) \qquad u_2 = -\frac{1}{t}$$

so $y_p = -\ln(t)\mathrm{e}^{-2t} - \mathrm{e}^{-2t}$. This last term is part of the homogeneous solution, so this simplifies to $-\ln(t)\mathrm{e}^{-2t}$. Now that we have all the parts,

$$y(t) = \mathrm{e}^{-2t}(C_1 + C_2t) - \ln(t)\mathrm{e}^{-2t}$$

You should note that here we have to use Variation of Parameters, since the forcing function is not one of the forms for Method of Undetermined Coefficients.

(b) $y'' - 2y' + y = t\mathrm{e}^t + 4$, $y(0) = 1$, $y'(0) = 1$.

With the Method of Undetermined Coefficients, we first get the homogeneous part of the solution,

$$y_h(t) = \mathrm{e}^t(C_1 + C_2t)$$

Now we construct our ansatz (Multiplied by t after comparing to y_h):

$$g_1 = te^t \quad \Rightarrow \quad y_{p_1} = (At + B)e^t \cdot t^2$$

Substitute this into the differential equation to solve for A, B:

$$y_{p_1} = (At^3 + Bt^2)e^t \qquad y'_{p_1} = (At^3 + (3A + B)t^2 + 2Bt)e^t$$

$$y''_{p_1} = (At^3 + (6A + B)t^2 + (6A + 4B)t + 2B)e^t$$

Forming $y''_{p_1} - 2y'_{p_1} + y_{p_1} = te^t$, we should see that $A = \frac{1}{6}$ and $B = 0$, so that $y_{p_1} = \frac{1}{6}t^3e^t$.
The next one is a lot easier! $y_{p_2} = A$, so $A = 4$, and:

$$y(t) = e^t(C_1 + C_2 t) + \frac{1}{6}t^3e^t + 4$$

with $y(0) = 1$, $C_1 = -3$. Solving for C_2 by differentiating should give $C_2 = 4$. The full solution:

$$y(t) = e^t\left(\frac{1}{6}t^3 + 4t - 3\right) + 4$$

(c) $y'' + 4y = 3\sin(2t)$, $y(0) = 2$, $y'(0) = -1$.
The homogeneous solution is $C_1\cos(2t) + C_2\sin(2t)$. Just for fun, you could try Variation of Parameters. We'll outline the Method of Undetermined Coefficients:

$$y_p = (A\sin(2t) + B\cos(2t))t = At\sin(2t) + Bt\cos(2t)$$

$$y''_p = (-4At - 4B)\sin(2t) + (4A - 4Bt)\cos(2t)$$

taking $y''_p + 4y_p = 3\sin(2t)$, we see that $A = 0$, $B = -\frac{3}{4}$, so the solution is:

$$y = c_1\cos(2t) + c_2\sin(2t) - \frac{3}{4}t\cos(2t)$$

With $y(0) = 2$, $c_1 = 2$. Differentiating to solve for c_2, we find that $c_2 = -1/8$.

(d) $y'' + 9y = \sum_{m=1}^{N} b_m\cos(m\pi t)$
The homogeneous part of the solution is $C_1\cos(3t) + C_2\sin(3t)$. We see that $3 \neq m\pi$ for $m = 1, 2, 3, \ldots$.
The forcing function is a sum of N functions, the m^{th} function is:

$$g_m(t) = b_m\cos(m\pi t) \quad \Rightarrow \quad y_{p_m} = A\cos(m\pi t) + B\sin(m\pi t)$$

Differentiating,

$$y''_{p_m} = -m^2\pi^2A\cos(m\pi t) - m^2\pi^2B\sin(m\pi t)$$

so that $y''_{p_m} + 9y_{p_m} = (9 - m^2\pi^2)A\cos(m\pi t) + (9 - m^2\pi^2)B\sin(m\pi t)$.
Solving for the coefficients, we see that $A = b_m/(9 - m^2\pi^2)$ and $B = 0$. Therefore, the full solution is:

$$y(t) = C_1\cos(3t) + C_2\sin(3t) + \sum_{m=1}^{N} \frac{b_m}{9 - m^2\pi^2}\cos(m\pi t)$$

13. Rewrite the expression in the form $a + ib$: (i) 2^{i-1} (ii) $e^{(3-2i)t}$ (iii) $e^{i\pi}$

NOTE for the SOLUTION: Remember that for any non-negative number A, we can write $A = e^{\ln(A)}$.

- $2^{i-1} = e^{ln(2^{i-1})} = e^{(i-1)\ln(2)} = e^{-\ln(2)}e^{i\ln(2)} = \frac{1}{2}\left(\cos(\ln(2)) + i\sin(\ln(2))\right)$
- $e^{(3-2i)t} = e^{3t}e^{-2ti} = e^{3t}\left(\cos(-2t) + i\sin(-2t)\right) = e^{3t}\left(\cos(2t) - i\sin(2t)\right)$
  (Recall that cosine is an even function, sine is an odd function).
- $e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1$

14. Write $a + ib$ in polar form: (i) $-1 - \sqrt{3}i$ (ii) $3i$ (iii) -4 (iv) $\sqrt{3} - i$

SOLUTIONS:

(i) $r = \sqrt{1+3} = 2$, $\theta = -2\pi/3$ (look at its graph, use 30-60-90 triangle):

$$-1 - \sqrt{3}i = 2e^{-\frac{2\pi}{3}i}$$

(ii) $3i = 3e^{\frac{\pi}{2}i}$

(iii) $-4 = 4e^{\pi i}$

(iv) $\sqrt{3} - i = 2e^{-\frac{\pi}{6}i}$

15. Find a second order linear differential equation with constant coefficients whose general solution is given by:

$$y(t) = C_1 + C_2e^{-t} + \frac{1}{2}t^2 - t$$

SOLUTION: Work backwards from the characteristic equation to build the homogeneous DE (then figure out the rest):

The roots to the characteristic equation are $r = 0$ and $r = -1$. The characteristic equation must be $r(r+1) = 0$ (or a constant multiple of that). Therefore, the differential equation is:

$$y'' + y' = 0$$

For $y_p = \frac{1}{2}t^2 - t$ to be the particular solution,

$$y''_p + y'_p = (1) + (t - 1) = t$$

so the full differential equation must be:

$$y'' + y' = t$$

16. Determine the longest interval for which the IVP is certain to have a unique solution (Do not solve the IVP):

$$t(t-4)y'' + 3ty' + 4y = 2 \qquad y(3) = 0 \quad y'(3) = -1$$

SOLUTION: Write in standard form first-

$$y'' + \frac{3}{t-4}y' + \frac{4}{t(t-4)}y = \frac{2}{t(t-4)}$$

The coefficient functions are all continuous on each of three intervals:

$$(-\infty, 0), (0, 4) \text{ and } (4, \infty)$$

Since the initial time is 3, we choose the middle interval, $(0, 4)$.

17. Let $L(y) = ay'' + by' + cy$ for some value(s) of a, b, c.
If $L(3e^{2t}) = -9e^{2t}$ and $L(t^2 + 3t) = 5t^2 + 3t - 16$, what is the particular solution to:

$$L(y) = -10t^2 - 6t + 32 + e^{2t}$$

SOLUTION: This purpose of this question is to see if we can use the properties of linearity to get at the answer.
We see that: $L(3e^{2t}) = -9e^{2t}$, or $L(e^{2t}) = -3e^{2t}$ so:

$$L\left(-\frac{1}{3}e^{2t}\right) = e^{2t}$$

And for the second part,

$$L(t^2 + 3t) = 5t^2 + 3t - 16 \quad \Rightarrow \quad L((-2)(t^2 + 3t)) = -10t^2 + 6t - 32$$

The particular solution is therefore:

$$y_p(t) = -2(t^2 + 3t) - \frac{1}{3}e^{2t}$$

since we have shown that

$$L\left(-2(t^2 + 3t) - \frac{1}{3}e^{2t}\right) = -10t^2 + 6t - 32 + e^{2t}$$

18. Solve the following Euler equations:

(a) $t^2y'' + 2ty' - 2y = 0$

SOLUTION: With the ansatz $y = t^r$, the characteristic equation becomes $r(r-1) + 2r - 2 = 0$, or

$$r^2 + r - 2 = 0 \quad \Rightarrow \quad (r+2)(r-1) = 0$$

Therefore,

$$y(t) = C_1 t + C_2 t^{-2}$$

(b) $t^2y'' + ty' + 9y = 0$

SOLUTION: Same as before, use the ansatz $y = t^r$ and the characteristic equation becomes

$$r(r-1) + r + 9 = 0 \quad \Rightarrow \quad r^2 + 9 = 0 \quad \Rightarrow \quad r = \pm 3i$$

How do we interpret t^{3i}? You can either recall the formula from class, or reconstruct it:

$$t^{3i} = \mathrm{e}^{3i\ln(t)} = \cos(3\ln(t)) + i\sin(3\ln(t))$$

So the two functions for the fundamental set are (no i):

$$y(t) = C_1\cos(3\ln(t)) + C_2\sin(3\ln(t))$$

19. Consider the DE: $u'' + u' + 2u = \cos(\omega t)$.

(a) Solve the homogeneous equation:

SOLUTION: Working with the characteristic equation, we get

$$r^2+r+2 = 0 \quad \Rightarrow \quad r^2+r+\frac{1}{4} = -2+\frac{1}{4} \quad \Rightarrow \quad \left(r+\frac{1}{2}\right)^2 = -\frac{7}{4} \quad \Rightarrow \quad r = -\frac{1}{2} \pm \frac{\sqrt{7}}{2}\,i$$

Therefore,

$$y_h = \mathrm{e}^{-t/2}\left(C_1\cos\left(\frac{\sqrt{7}}{2}t\right) + C_2\sin\left(\frac{\sqrt{7}}{2}t\right)\right)$$

Notice that this part of the overall solution goes to zero for any choice of constants C_1, C_2 (this is called the transient part of the solution for that reason).

(b) If the amplitude of the particular part of the solution (also called the steady state part) is given by

$$R = \frac{1}{\sqrt{\omega^4 - 3\omega^2 + 4}}$$

find the critical point for R:

SOLUTION: Taking the derivative with respect to ω and setting it to zero:

$$\frac{dR}{d\omega} = -\frac{1}{2}(\omega^4 - 3\omega^2 + 4)^{-3/2}(4\omega^3 - 6\omega) = -\frac{4\omega^3 - 6\omega}{2(\omega^4 - 3\omega^2 + 4)} = 0$$

Clear the denominator (it will never be zero), and continue solving:

$$4\omega^3 - 6\omega = 0 \quad \Rightarrow \quad 2\omega(2\omega^2 - 3) = 0 \quad \Rightarrow \quad \omega = 0, \quad \omega = \pm\sqrt{\frac{3}{2}}$$

We take the positive solution, $\omega = \sqrt{3/2}$.

(c) For part (c), we note that this value of $\omega \approx 1.2$ while $\sqrt{2} \approx 1.4$. If we were to put in a damping constant closer to 0 than 1 (for example, $\gamma = \frac{1}{10}$), then our value of ω would be a lot closer to $\sqrt{2}$.

20. Explain in words the concepts of beating and resonance and how they relate to each other.

SOLUTION: Given $mu'' + ku = F_0\cos(\omega t)$, the (circular) frequency of the homogeneous part of the solution is $\omega_0 = \sqrt{k/m}$. If the forcing frequency is changed so that $|\omega - \omega_0|$ is small, then "beating" occurs- That is where the particular solution is a product of two periodic functions, one with a long period and one with a very short period.

As ω gets closer and closer to ω_0, the amplitude and period of the particular part of the solution begin to get larger and larger, until finally at $\omega = \omega_0$, the particular part of the solution "explodes", or becomes unbounded. We see that in the choice for the particular solution from the method of undetermined coefficients:

$$y_p = t(A\cos(\omega_0 t) + B\sin(\omega_0 t))$$

21. Use Variation of Parameters to find a particular solution to the following, then verify your answer using the Method of Undetermined Coefficients:

$$4y'' - 4y' + y = 16\mathrm{e}^{t/2}$$

SOLUTION: For the Variation of Parameters, write in standard form first, then compute $y_1, y_2, W(y_1, y_2)$, and then use the formulas (they will be given to you):

$$y'' - y' + \frac{1}{4}y = 4\mathrm{e}^{t/2} \quad \Rightarrow \quad r^2 - r + \frac{1}{4} = 0 \quad \Rightarrow \quad r = 1/2, 1/2$$

Therefore,

$$y_1 = \mathrm{e}^{t/2} \qquad y_2 = t\mathrm{e}^{t/2} \qquad W(y_1, y_2) = \mathrm{e}^t$$

And,

$$u_1' = \frac{-4t\mathrm{e}^t}{\mathrm{e}^t} = -4t \qquad u_2' = \frac{4\mathrm{e}^t}{\mathrm{e}^t}$$

so that $u_1 = -2t^2$ and $u_2 = 4t$. Putting these back into our ansatz,

$$y_p = -2t^2\mathrm{e}^{t/2} + 4t(t^2\mathrm{e}^{t/2}) = 2t^2\mathrm{e}^{t/2}$$

You can verify this solution using the Method of Undetermined Coefficients.

22. Compute the Wronskian of two solutions of the given DE without solving it:

$$x^2y'' + xy' + (x^2 - \alpha^2)y = 0$$

Using Abel's Theorem (and writing the DE in standard form first):

$$y'' + \frac{1}{x}y' + \frac{x^2 - \alpha^2}{x^2}y = 0$$

Therefore,

$$W(y_1, y_2) = C\mathrm{e}^{-\int \frac{1}{x}\,dx} = \frac{C}{x}$$

23. If $y'' - y' - 6y = 0$, with $y(0) = 1$ and $y'(0) = \alpha$, determine the value(s) of α so that the solution tends to zero as $t \to \infty$.

SOLUTION: Solving as usual gives:

$$y(t) = \left(\frac{3 - \alpha}{5}\right)\mathrm{e}^{-2t} + \left(\frac{\alpha + 2}{5}\right)\mathrm{e}^{3t}$$

so to make sure the solutions tend to zero, $\alpha = -2$ (to zero out the second term).

24. Give the general solution to $y'' + y = \frac{1}{\sin(t)} + t$

SOLUTION: If we can use any method, I would use Method of Undetermined Coefficients for $g_2(t) = t$ and Variation of Parameters for $g_1(t) = 1/\sin(t)$. Let's do the Variation of Parameters first:

$$y_1 = \cos(t) \qquad y_2 = \sin(t) \qquad W(y_1, y_2) = 1 \qquad g(t) = \frac{1}{\sin(t)}$$

Therefore,

$$u_1' = -1 \qquad u_2' = \frac{\cos(t)}{\sin(t)}$$

Continuing,

$$u_1 = -t \qquad u_2 = \ln(\sin(t))$$

and

$$y_p = -t\cos(t) + \ln(\sin(t))\sin(t)$$

For the other part, we take $y_p = At + B$, and find that $y_p = t$, so that

$$y(t) = C_1\cos(t) + C_2\sin(t) + t - t\cos(t) + \ln(\sin(t))\sin(t)$$

25. A mass of 0.5 kg stretches a spring to 0.05 meters. (i) Find the spring constant. (ii) Does a stiff spring have a large spring constant or a small spring constant (explain).

SOLUTION: (Typo: Better to read this as stretching the spring an additional 0.05 meters- Otherwise, we wouldn't be able to compute L).

We use Hooke's Law at equilibrium: $mg - kL = 0$, or

$$k = \frac{mg}{L} = \frac{4.9}{0.05} = 98$$

For the second part, a stiff spring will not stretch, so L will be small (and k would therefore be large), and a spring that is not stiff will stretch a great deal (so that k will be smaller).

26. A mass of 5 kg stretches a spring 0.1 m. The mass is acted on by an external force of $10sin(t/2)$ N (newtons) and moves in a medium that imparts a viscous force of 2 N when the speed of the mass is 0.04 meters per second. If the mass is set in motion by pulling down 0.3 meters and imparting a velocity of 0.03 meters per second, formulate the initial value problem describing the motion of the mass (do not solve).

SOLUTION: The mass is 5 kg and stretches the spring 0.1 meters means that:

$$mg - kL = 0 \quad \Rightarrow \quad k = \frac{mg}{L} = \frac{5 \cdot 9.8}{0.1} = \frac{5 \cdot 9.8 \cdot 10}{1} = 490$$

For the damping constant, our assumption is that the damping force is proportional to the velocity. Therefore, if F_d is the damping force, then:

$$F_d = 2 \qquad \text{and} \qquad F_d = \gamma u' \quad \Rightarrow \quad 2 = \gamma \cdot 0.04 \quad \Rightarrow \quad \gamma = 50$$

Finally, since we set the weight mg to be positive in our model, $u(t) > 0$ when the mass is below equilibrium. I should have been more specific about the direction of the velocity- Set it to positive. Therefore,

$$5u'' + 50u' + 490u = 10\left(\frac{1}{2}t\right) \qquad u(0) = 0.3 \quad u'(0) = 0.03$$

27. Match the solution curve to its IVP (There is one DE with no graph, and one graph with no DE- You should not try to completely solve each DE).

   (a) $5y'' + y' + 5y = 0$, $y(0) = 10$, $y'(0) = 0$ (Complex roots, solutions go to zero) Graph C

   (b) $y'' + 5y' + y = 0$, $y(0) = 10$, $y'(0) = 0$ (Exponentials, solutions go to zero) Graph D

   (c) $y'' + y' + \frac{5}{4}y = 0$, $y(0) = 10$, $y'(0) = 0$ NOT USED

(d) $5y'' + 5y = 4\cos(t)$, $y(0) = 0$, $y'(0) = 0$ (Pure Harmonic) Graph B

(e) $y'' + \frac{1}{2}y' + 2y = 10$, $y(0) = 0$, $y'(0) = 0$ (Complex roots to homogeneous solution, constant particular solution) Graph E

SOLUTION: If the graphs are labeled: Top row: A, B, second row: C, D, and last row E, then the graphs are given above.