

Example: Algebra on the Index

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$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n$$

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Simplify to one sum that uses the term x^n :

$$\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} + x \sum_{k=1}^{\infty} k a_k x^{k-1} = \sum_{n=?}^? (C_n) x^n$$

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SOLUTION: Try writing out the first few terms:

$$\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} = 2a_2 + 2 \cdot 3a_3 x + 3 \cdot 4a_4 x^2 + 4 \cdot 5a_5 x^3 + \dots$$

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$$\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} + x \sum_{k=1}^{\infty} k a_k x^{k-1} = \sum_{n=?}^? \binom{?}{C_n} x^n$$

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Powers of x don't line up: *To write this as a single sum, we need to manipulate the sums so that the powers of x line up.*

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Substitute $n = m - 2$ (or $m = n + 2$) into the first sum, and $n = k$ into the second sum:

$$\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n$$

$$\sum_{k=0}^{\infty} ka_k x^k = \sum_{n=0}^{\infty} na_n x^n$$

Finishing the solution:

$$\begin{aligned} & \sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} + x \sum_{k=1}^{\infty} k a_k x^{k-1} \\ &= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + \sum_{n=0}^{\infty} n a_n x^n \end{aligned}$$

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Alternate Solution:

We could have started both indices using x^1 instead of x^0 . Here are the sums again:

$$\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} = 2a_2 + [3 \cdot 2a_3x + 4 \cdot 3a_4x^2 + 5 \cdot 4a_5x^3 + \dots]$$

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In this case,

$$\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} = 2a_2 + \sum_{m=3}^{\infty} m(m-1)a_m x^{m-2}$$

and let $n = m - 2$ to get

$$2a_2 + \sum_{n=1}^{\infty} [(n+2)(n+1)a_{n+2} + na_n] x^n$$

Using Series in DEs

Given $y'' + p(x)y' + q(x)y = 0$, $y(x_0) = y_0$ and $y'(x_0) = v_0$, assume y, p, q are analytic at x_0 .

Ansatz:

$$y(t) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

so that

$$y'(t) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1} \quad \text{and} \quad y''(t) = \sum_{n=2}^{\infty} n(n-1) a_n (x - x_0)^{n-2}$$

The Big Picture

Substituting the series into the DE will give something like:

$$\sum(\dots) + \sum(\dots) + \sum(\dots) = 0$$

We will want to write this in the form:

$$\sum (C_n) x^n = 0$$

Then we will set $C_n = 0$ for each n .

Example: Power Series into a DE

Find the recurrence relation and the first four terms of a fundamental set of solutions to:

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Substitute $y = \sum_{n=0}^{\infty} a_n(x-1)^n$:

$$\sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} - x \sum_{n=1}^{\infty} na_n(x-1)^{n-1} - \sum_{n=0}^{\infty} a_n(x-1)^n = 0$$

Look at/Simplify the second term:

$$-x \sum_{n=1}^{\infty} na_n(x-1)^{n-1} =$$

$$-x \sum_{n=1}^{\infty} na_n(x-1)^{n-1} + 1 \sum_{n=1}^{\infty} na_n(x-1)^{n-1} - 1 \sum_{n=1}^{\infty} na_n(x-1)^{n-1}$$

to get something useful:

$$- \sum_{n=1}^{\infty} na_n(x-1)^n - \sum_{n=1}^{\infty} na_n(x-1)^{n-1}$$

We now have 4 sums:

$$\sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} - \sum_{n=1}^{\infty} na_n(x-1)^n$$
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Ready to simplify? (Check powers)

Reset the indices:

$$\sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} - \sum_{n=0}^{\infty} na_n(x-1)^n$$
$$- \sum_{n=1}^{\infty} na_n(x-1)^{n-1} - \sum_{n=0}^{\infty} a_n(x-1)^n = 0$$

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Make the substitutions:

$k = n - 2$	$k = n$	$k = n - 1$	$k = n$
$n = k + 2$	$n = k$	$n = k + 1$	$n = k$

Reset the indices:

$$\begin{aligned} \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} - \sum_{n=0}^{\infty} na_n(x-1)^n \\ - \sum_{n=1}^{\infty} na_n(x-1)^{n-1} - \sum_{n=0}^{\infty} a_n(x-1)^n = 0 \end{aligned}$$

Make the substitutions:

$$\begin{array}{cccc} k = n - 2 & k = n & k = n - 1 & k = n \\ n = k + 2 & n = k & n = k + 1 & n = k \end{array}$$

To get:

$$\sum_{k=0}^{\infty} ((k+2)(k+1)a_{k+2} - ka_k - (k+1)a_{k+1} - a_k)(x-1)^k = 0$$

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means:

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Solve for a_{k+2} :

$$a_{k+2} = \frac{1}{k+2} (a_k + a_{k+1})$$

This is the recurrence relation.

Build up a solution from the recurrence

Find y_1, y_2 that solve:

$$y(1) = a_0 = 1 \quad y'(1) = a_1 = 0 \quad y(1) = a_0 = 0 \quad y'(1) = a_1 = 1$$

(Therefore, $W(y_1, y_2)(1) = 1$)

We do this for y_1 in the first column, y_2 in the second:

$$k = 0 \quad a_2 = \frac{1}{2}(a_0 + a_1) = \frac{1}{2}$$

$$k = 1 \quad a_3 = \frac{1}{3}(a_1 + a_2) = \frac{1}{6}$$

$$k = 2 \quad a_4 = \frac{1}{4}(a_2 + a_3) = \frac{1}{6}$$

$$k = 3 \quad a_5 = \frac{1}{5}(a_3 + a_4) = \frac{1}{15}$$

and

$$k = 0 \quad a_2 = \frac{1}{2}(a_0 + a_1) = \frac{1}{2}$$

$$k = 1 \quad a_3 = \frac{1}{3}(a_1 + a_2) = \frac{1}{2}$$

$$k = 2 \quad a_4 = \frac{1}{4}(a_2 + a_3) = \frac{1}{4}$$

$$k = 3 \quad a_5 = \frac{1}{5}(a_3 + a_4) = \frac{3}{20}$$

We now have two linearly independent solutions to the DE:

$$y_1(x) = 1 + \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 + \frac{1}{6}(x-1)^4 + \dots$$

and

$$y_2(x) = (x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{2}(x-1)^3 + \frac{1}{4}(x-1)^4 + \dots$$