Complex Integrals and the Laplace Transform

There are a few computations for which the complex exponential is very nice to use. We'll see a few here, but first a couple of Theorems about integrating a complex function:

Theorem: $\int \mathrm{e}^{(a+bi)t}\,dt = \frac{1}{(a+bi)}\mathrm{e}^{(a+bi)t}$
The proof is to just work this out using Euler's formula- It's a nice exercise when you have a little time (you'll need to do integration by parts twice).

Theorem: The Laplace Transform of $\mathrm{e}^{(a+ib)t}$:

$$\mathcal{L}(\mathrm{e}^{(a+ib)t}) = \frac{1}{s-(a+ib)}$$

The proof relies on showing the work in the first exercise.

Theorem: More generally, we can use the previous "trick" on any integral that can be written in complex exponential form. Two examples:

$$\int \mathrm{e}^{at}\cos(bt)\,dt = \text{Real}\left(\frac{1}{a+ib}\mathrm{e}^{(a+ib)t}\right)$$
$$\int \mathrm{e}^{at}\sin(bt)\,dt = \text{Imag}\left(\frac{1}{a+ib}\mathrm{e}^{(a+ib)t}\right)$$

Worked Example:

1. Use complex exponentials to compute $\int \mathrm{e}^{2t}\cos(3t)\,dt$.

   SOLUTION: We note that $\mathrm{e}^{2t}\cos(3t) = \text{Real}(\mathrm{e}^{(2+3i)t})$, so:

   $$\int \mathrm{e}^{2t}\cos(3t)\,dt = \text{Real}\left(\frac{1}{2+3i}\mathrm{e}^{(2+3i)t}\right)$$

   Simplifying the term inside the parentheses and multiplying out the complex terms:

   $$\mathrm{e}^{2t}\left(\frac{2-3i}{4+9}\right)(\cos(3t)+i\sin(3t)) =$$

   $$\mathrm{e}^{2t}\left[\left(\frac{2}{13}\cos(3t)+\frac{3}{13}\sin(3t)\right)+i\left(-\frac{3}{13}\cos(3t)+\frac{2}{13}\sin(3t)\right)\right]$$

   Therefore,
   $$\int \mathrm{e}^{2t}\cos(3t)\,dt = \mathrm{e}^{2t}\left(\frac{2}{13}\cos(3t)+\frac{3}{13}\sin(3t)\right)$$

   In fact, we get the other integral for free:

   $$\int \mathrm{e}^{2t}\sin(3t)\,dt = \mathrm{e}^{2t}\left(\frac{-3}{13}\cos(3t)+\frac{2}{13}\sin(3t)\right)$$

2. Use complex exponentials to compute the Laplace transform of $\cos(at)$:

   SOLUTION: Note that $\cos(at) = \text{Real}(\mathrm{e}^{(at)i})$

$$\mathcal{L}(\cos(at) + i\sin(at)) = \mathcal{L}(\mathrm{e}^{ait}) = \frac{1}{s - ai} = \frac{s + ai}{s^2 + a^2}$$

   so the real part gives us the Laplace transform of $\cos(at)$ and the imaginary part gives the Laplace transform of $\sin(at)$:

$$\mathcal{L}(\cos(at)) = \frac{s}{s^2 + a^2} \qquad \mathcal{L}(\sin(at)) = \frac{a}{s^2 + s^2}$$

Homework Addition to Section 6.1

1. Use Euler's Formula to show that, if a, b are constants, and s is a parameter, then

$$\lim_{t\to\infty} \mathrm{e}^{-(s-(a+ib))t} = 0 \qquad for s > a$$

2. Use complex exponentials to compute $\int \mathrm{e}^{-2t}\sin(3t)\,dt$.

3. Use complex exponentials to compute the Laplace transform of $\sin(at)$.

4. Use complex exponentials to compute the Laplace transform of $\mathrm{e}^{at}\sin(bt)$ and $\mathrm{e}^{at}\cos(bt)$ (compare to exercises 13, 14).

5. Prove that e^t goes to infinity faster than any polynomial. You can do that by showing

$$\lim_{t\to\infty} \frac{t^n}{\mathrm{e}^t} = 0$$

6. (**The Racetrack Principle**) We can show that $f(x) < g(x)$ for all $x \geq a$ by proving two things: (i) $f(a) < g(a)$, and (ii) $f'(x) < g'(x)$ for all $x > a$. Use this idea to prove that $\ln(t) < t$ for all $t \geq 1$ (it is true for all $t > 0$, but we wouldn't be able to use this argument for $0 < t < 1$).

7. Show that, if $f(t)$ is bounded (that is, there is a constant A so that $|f(t)| \leq A$ for all t), then f is of exponential order (do this by finding K, a and M from the definition).

8. If the function is of exponential order, find the K, a and M from the definition. Otherwise, state that it is not of exponential order.

   Something that may be handy from algebra: $A = \mathrm{e}^{\ln(A)}$.

   (a) $\sin(t)$
   (b) $\tan(t)$
   (c) t^3
   (d) e^{t^2}
   (e) 5^t
   (f) t^t

9. Use complex exponentials to find the Laplace transform of $t\sin(at)$.