SOLUTIONS to the exercises in the complex Laplace handout

1. Use Euler's Formula to show that, if a, b are constants, and s is a parameters, then
$$\lim_{t\to\infty} \mathrm{e}^{-(s-(a+ib))t} = 0$$
SOLUTION: First write out the expression.
$$\mathrm{e}^{-(s-(a+ib))t} = \mathrm{e}^{-(s-a)t}\mathrm{e}^{-i(bt)}$$
Since e^{-ibt} is on the unit circle, its magnitude is 1. Therefore, the magnitude of the exponential function is simply:
$$|\mathrm{e}^{-(s-(a+ib))t}| = \mathrm{e}^{-(s-a)t}$$
and as long as $s - a > 0$, the magnitude will go to zero as $t \to \infty$, which means the original function must also go to zero.

2. Use complex exponentials to compute $\int \mathrm{e}^{-2t}\sin(3t)\,dt$.

   SOLUTION: We'll find the imaginary part of $\int \mathrm{e}^{(-2+3i)t}\,dt$:
$$\int \mathrm{e}^{(-2+3i)t}\,dt = \frac{1}{-2+3i}\mathrm{e}^{(-2+3i)t} = \mathrm{e}^{-2t}(\cos(3t) + i\sin(3t)) \cdot \frac{-2-3i}{4+9} =$$
$$\frac{\mathrm{e}^{-2t}}{13}((-2\cos(3t) + 3\sin(3t)) + i(-3\cos(3t) - 2\sin(3t))$$
The imaginary part gives us the answer:
$$\int \mathrm{e}^{-2t}\sin(3t)\,dt = \frac{\mathrm{e}^{-2t}}{13}(-3\cos(2t) - 2\sin(3t))$$

3. Use complex exponentials to compute the Laplace transform of $\sin(at)$.

   SOLUTION: Use $\cos(at) + i\sin(at) = \mathrm{e}^{iat}$. Then we'll compute the transform of e^{iat} using the table and pull off the imaginary part.
$$\mathcal{L}(\mathrm{e}^{iat}) = \frac{1}{s-ai} = \frac{s+ia}{s^2+a^2}$$
Therefore, the imaginary part is: $\frac{a}{s^2+a^2}$

4. Use complex exponentials to compute the Laplace transform of $\mathrm{e}^{at}\sin(bt)$ and $\mathrm{e}^{at}\cos(bt)$.

   SOLUTION: Use the fact that $\mathrm{e}^{(a+ib)t}$ gives us both expressions, and
$$\mathcal{L}(\mathrm{e}^{(a+ib)t} = \frac{1}{s-(a+ib)} = \frac{1}{(s-a)-ib} = \frac{(s-a)+ib}{(s-a)^2+b^2}$$
Therefore, we get both transforms:
$$\mathcal{L}(\mathrm{e}^{at}\cos(bt)) = \frac{s-a}{(s-a)^2+b^2} \qquad \mathcal{L}(\mathrm{e}^{at}\sin(bt)) = \frac{b}{(s-a)^2+b^2}$$

5. Prove that e^t goes to infinity faster than any polynomial by considering the limit below.

   SOLUTION: Use l'Hospital's rule n times to get:

$$\lim_{t\to\infty}\frac{t^n}{e^t}=\lim_{t\to\infty}\frac{nt^{n-1}}{e^t}=\lim_{t\to\infty}\frac{n(n-1)t^{n-2}}{e^t}=\cdots=\lim_{t\to\infty}\frac{n!}{e^t}=0$$

6. Use the Racetrack Principle to show that, for $t \geq 1$, $\ln(t) < t$.

   SOLUTION: Let $f(t) = \ln(t)$ and $g(t) = t$. Then:

   - $f(1) = 0$ and $g(1) = 1$, so $f(1) < g(1)$.
   - $f'(x) = 1/x$ and $g'(x) = 1$, so for $x > 1$, $f'(x) < g'(x)$.

   Therefore, by the Racetrack Principle, $\ln(t) < t$ for all $t \geq 1$.

7. If $f(t)$ is bounded by a number A, then

$$|f(t)| \leq A = Ke^{at}, \quad t \geq M.$$

   So f is of exponential order with $K = A, a = 0$ and M anything.

8. Exponential order practice:

   (a) $\sin(t)$: Since $|\sin(t)| \leq 1$, we can do the same thing as in the previous exercise with $A = 1$.

   (b) $\tan(t)$ has a vertical asymptote at $t = \pi/2$ (and multiples of π thereafter), so it is NOT of exponential order.

   (c) $t^3 = e^{\ln(t^3)} = e^{3\ln(t)} \leq e^{3t}$, so this is of exponential order.

   (d) e^{t^2} is not of exponential order, since the exponent is of order 2.

   (e) $5^t = e^{\ln(5^t)} = e^{t\ln(5)}$

   (f) t^t is not of exponential order:

$$t^t = e^{\ln(t^t)} = e^{t\ln(t)}$$