Last time:

- Vocab: ODE, PDE, IVP
- Skills: Be able to verify that $\phi(t)$ is a solution to a DE.
- Given the general solution to $y' = ay + b$ (it is $y = Pe^{at} - \frac{b}{a}$)
- Three models: Free fall, Mice/Owls, Newton's Law of Cooling.

Today: Finish up visualizations in Chapter 1, look at an algorithm in 2.1. First, let's get a solution to $y' = ay + b$. Notice that this DE could be expressed as:

$$\left(y + \frac{b}{a}\right)' = a\left(y + \frac{b}{a}\right)$$

which is the normal exponential growth model. That is, if $Y = y + b/a$, then the DE is

$$Y' = aY \quad \Rightarrow \quad Y = C\mathrm{e}^{at}$$

or

$$y + \frac{b}{a} = C\mathrm{e}^{at} \quad \Rightarrow \quad y = C\mathrm{e}^{at} - \frac{b}{a}$$

where C depends on the initial conditions...

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SOLUTION:

$$y(t) = Ce^{-2t} + \frac{5}{2}$$

We note that for any C, the solution will converge to $5/2$ as t gets large.

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  (More on this later)

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TODAY: Visualizing solutions, solving a linear equation.

Visualizing solutions to DE

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$$y' = ay + b \qquad y(t) = P_0 e^{at} - \frac{b}{a}$$

Cases:

- If $P_0 = 0$, then $y(t)$ is constant ($y = -b/a$).
  Definition: An **equilibrium solution** is a constant solution $y = k$ so that $y' = 0$.
- Otherwise:
  - If $a > 0$, then the solutions will all "blow up" ($|y(t)| \to \infty$) except one solution.
  - If $a < 0$, then all solutions tend toward equilibrium.

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Example: $y' = t - y^2$
In drawing a picture, we might consider curves of constant slope. For example, with zero slope:

$$0 = t - y^2 \quad \Rightarrow \quad y^2 = t$$

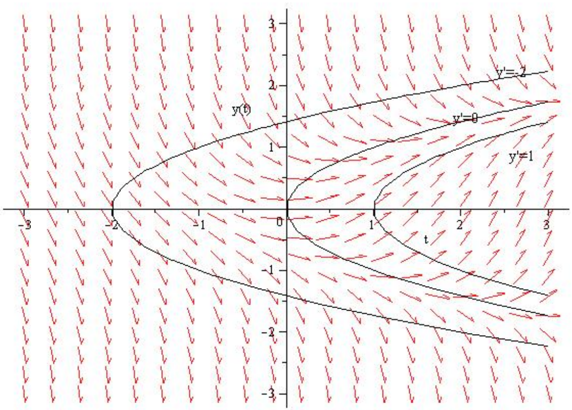
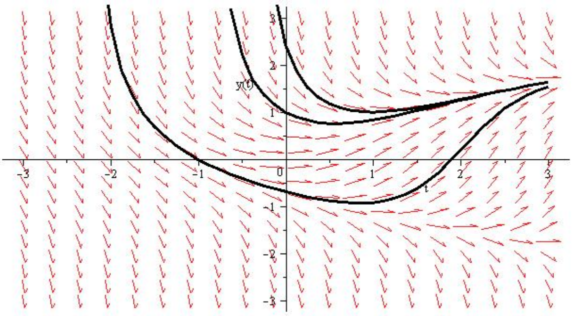
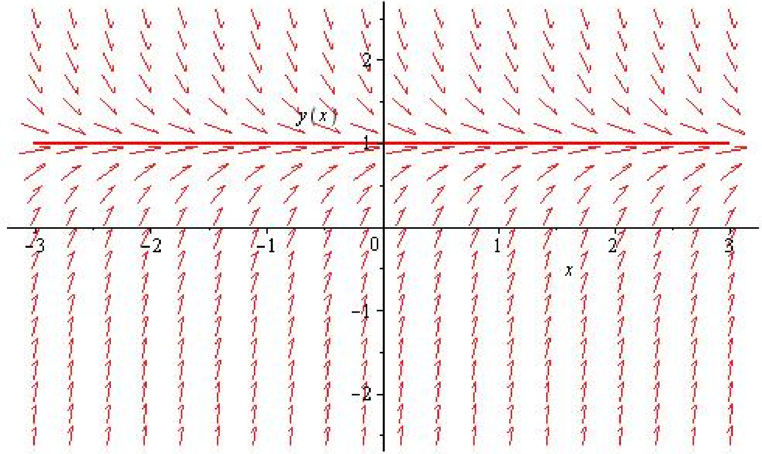


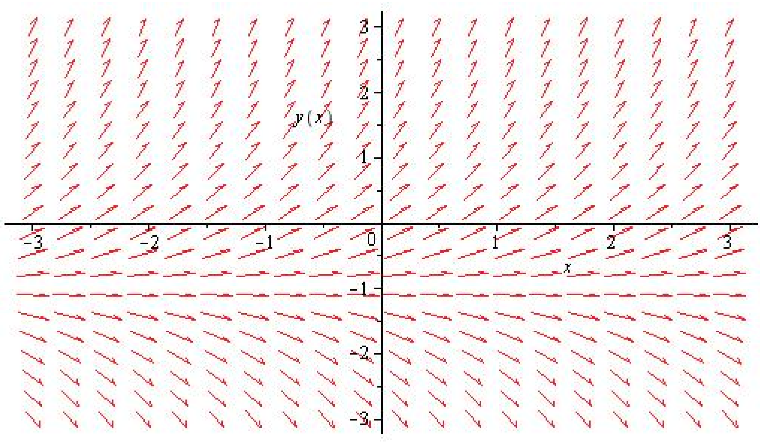
Figure : Direction Field with Isoclines: $y' = -2, y' = 0, y' = 1$


y(t)
t
-3
-2
-1
0
1
2
3

Give an ODE of the form $y' = ay + b$ whose direction field looks like:

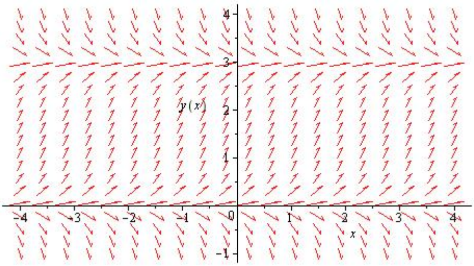


Same question as before:



Choose a DE

1. $y' = 3 - y$
2. $y' = y(y + 3)$
3. $y' = y(3 - y)$
4. $y' = 2y - 1$



Homework Hint: #22, Section 1.1

$$V = \frac{4}{3}\pi r^3 \qquad A = 4\pi r^2$$

so if $V' = kA$, give V' in terms of V only.

Homework Hint: #14, Section 1.3

Differentiate the following with respect to t:

$$f(t)\int_0^t G(s)\,ds$$

SOLUTION: Use the product rule and the FTC:

$$f'(t)\int_0^t G(s)\,ds + f(t)G(t)$$

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Question: Is there a function $e^{P(t)}$ that will turn the left side of the DE to the derivative of something?

Solve Linear DEs using Integrating Factor

Given $y' + a(t)y = f(t)$, we compute the **integrating factor**

$$e^{\int a(t)\,dt}$$

and multiply the DE by it:

$$e^{\int a(t)\,dt}(y' + a(t)y) = f(t)e^{\int a(t)\,dt}$$

This makes the left side a single derivative:

$$\left(y(t)e^{\int a(t)\,dt}\right)' = f(t)e^{\int a(t)\,dt}$$

which can be solved by integrating both sides.

$$y(t)e^{\int a(t)\,dt} = \int f(t)e^{\int a(t)\,dt}\,dt$$

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The general solution:

$$y = -\frac{1}{2}\mathrm{e}^{-2t} - \frac{1}{4t}\mathrm{e}^{-2t} + \frac{C}{t}$$