Last time: The Method of Undetermined Coefficients

$g_i(t)$ is :	The ansatz for y_{p_i} :
$P_n(t)$	$t^s(a_n t^n + \ldots + a_2 t^2 + a_1 t + a_0)$
$P_n(t)\mathrm{e}^{\alpha t}$	$t^s \mathrm{e}^{\alpha t}(a_n t^n + \ldots + a_2 t^2 + a_1 t + a_0)$
$P_n(t)\mathrm{e}^{\alpha t} \begin{cases} \sin(\beta t) \\ \cos(\beta t) \end{cases}$	$t^s \mathrm{e}^{\alpha t}\cos(\beta t)(a_n t^n + \ldots + a_2 t^2 + a_1 t + a_0) +$ $t^s \mathrm{e}^{\alpha t}\sin(\beta t)(b_n t^n + \ldots + b_2 t^2 + b_1 t + b_0)$

where $s = 0, 1$ or 2 so that no part of y_{p_i} is part of the homogeneous solution.

Example 1

Solve $y'' - 3y' - 4y = 3t\mathrm{e}^{-t}$
SOLUTION:

- Homogeneous part first: $(r-4)(r+1) = 0$, so

$$y_h(t) = C_1\mathrm{e}^{4t} + C_2\mathrm{e}^{-t}$$

- Particular part:
  First guess is: $(At + B)\mathrm{e}^{-t}$.
  Substituting back into the DE (algebra redacted):

$$-5A\mathrm{e}^{-t} = 3t\mathrm{e}^{-t} \quad \Rightarrow \quad -5A = 3t \text{ for all t}$$

  CORRECT GUESS: $y_p(t) = t(At + B)\mathrm{e}^{-t}$
  Note that $t^2\mathrm{e}^{-t}$ and $t\mathrm{e}^{-t}$ are not fcns in y_h.

If we continue with $y_p = (At^2 + Bt)\mathrm{e}^{-t}$, then substitute into the DE, and divide by e^{-t}, we get the following equation, *true for all t*:

$$-(10At + (5B - 2A)) = 3t$$

So we equate the coefficients and solve:

$$\begin{array}{lrl} t \text{ terms:} & -10A & = 3 \\ \text{Constants:} & 2A - 5B & = 0 \end{array} \quad \Rightarrow \quad \begin{array}{rl} A & = -3/10 \\ B & = -3/25 \end{array}$$

Put it all together at the end. The general solution is

$$y(t) = C_1\mathrm{e}^{4t} + \mathrm{e}^{-t}\left(C_2 - \frac{3}{25}t - \frac{3}{10}t^2\right)$$

Example 2: Give the ansatz for y_p (do not solve):

- $y'' - 3y' - 4y = 3t\cos(2t) + \sin(2t)$, with $r = -1, 4$.

$$y_p(t) = (At + B)\cos(2t) + (Ct + D)\sin(2t)$$

- $y'' - 3y' - 4y = \cos(3t) + t\sin(t)$, with $r = -1, 4$.

$$y_p(t) = A\cos(3t) + B\sin(3t) + (Ct + D)\cos(t) + (Et + F)\sin(t)$$

- $y'' - 2y' + 10y = t\mathrm{e}^t\sin(3t)$ with $r = 1 \pm 3i$

$$y_p(t) = \mathrm{e}^t\left[(At + B)\sin(3t) + (Ct + D)\cos(3t)\right]t$$

Give the form of y_p in each case:

- $y'' - 4y' + 4y = 3t^2$, with $r = -2, -2$.

$$y_p = At^2 + Bt + C$$

- $y'' + 4y = t^3 \mathrm{e}^t$ with $r = \pm 2i$

$$y_p = \mathrm{e}^t (At^2 + Bt + C)$$

- $y'' + 4y = 7t\cos(2t)$ with $r = \pm 2i$

$$y_p = t(At + B)\cos(2t) + (Ct + D)\sin(2t)$$

- $y'' - 4y' + 4y = \mathrm{e}^{-2t}$ with $r = -2, -2$.

$$y_p = t^2 A\mathrm{e}^{-2t}$$

3.6: Variation of Parameters

We're looking for a technique for getting the particular solution to the *GENERAL* linear second degree ODE:

$$y'' + p(t)y' + q(t)y = g(t) \qquad (*)$$

IDEA: Let y_1, y_2 be a fundamental set of solutions to $(*)$.
ANSATZ:

$$y_p = u_1(t)y_1(t) + u_2(t)y_2(t) = u_1y_1 + u_2y_2$$

so that:

$$y_p' = (u_1'y_1 + u_1y_1') + (u_2'y_2 + u_2y_2')$$

We will assume the following, which becomes one of our conditions:

$$u_1'y_1 + u_2'y_2 = 0$$

With that, we have:

$$
\begin{aligned}
y_p &= u_1 y_1 + u_2 y_2 \\
y_p' &= u_1 y_1' + u_2 y_2' \\
y_p'' &= u_1' y_1' + u_1 y_1'' + u_2' y_2 + u_2 y_2''
\end{aligned}
$$

Substitute into the DE: Substitute into the DE:

$$
\begin{array}{rlllll}
y_p'' & = u_1' y_1' + & u_1 y_1'' + & u_2' y_2 + & u_2 y_2'' \\
+p(t) y_p' & = & p u_1 y_1' + & & p u_2 y_2' \\
+q(t) y_p & = & q u_1 y_1 + & & q u_2 y_2 \\
\hline
g(t) & = u_1' y_1' + & 0 + & u_2' y_2' + & 0
\end{array}
$$

Summary Page: Variation of Parameters

Given $y'' + p(t)y' + q(t)y = g(t)$, let y_1, y_2 solve the homogeneous equation, and set

$$y_p = u_1 y_1 + u_2 y_2$$

Then u_1, u_2 satisfy the following equations

$$\begin{aligned} u_1' y_1 + u_2' y_2 &= 0 \\ u_1' y_1' + u_2' y_2' &= g(t) \end{aligned}$$

Which is solved via Cramer's Rule:

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ g(t) & y_2' \end{vmatrix}}{W(y_1, y_2)} \quad u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & g(t) \end{vmatrix}}{W(y_1, y_2)}$$

Cramer's Rule gave:

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ g(t) & y_2' \end{vmatrix}}{W(y_1, y_2)} \quad u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & g(t) \end{vmatrix}}{W(y_1, y_2)}$$

Which gives us u_1, u_2

$$u_1 = \int \frac{-y_2 g(t)}{W(y_1, y_2)}\, dt \quad u_2 = \int \frac{y_1 g(t)}{W(y_1, y_2)}\, dt$$

So that, if we wanted to write $y_p(t)$, we could:

$$y_p(t) = y_1(t) \int \frac{-y_2 g(t)}{W(y_1, y_2)}\, dt + y_2(t) \int \frac{y_1 g(t)}{W(y_1, y_2)}\, dt$$

Use any method to find the general solution:

$$y'' - 2y' + y = \frac{\mathrm{e}^t}{1+t^2} + t^2\mathrm{e}^t$$

SOLUTION: First, the solution to the characteristic equation is $r = 1, 1$:

$$y_h = \mathrm{e}^t(C_1 + C_2 t)$$

And...?

For $g_1(t) = \mathrm{e}^t/(1+t^2)$, use Variation of Parameters:

$$y_1 = \mathrm{e}^t, \quad y_2 = t\mathrm{e}^t, \quad W = \mathrm{e}^{2t}$$

$$u_1' = \frac{-y_2 g}{W} = \frac{-t\mathrm{e}^t \frac{\mathrm{e}^t}{(1+t^2)}}{\mathrm{e}^{2t}} = \frac{-t}{1+t^2} \qquad u_2' = \frac{y_1 g}{W} = \frac{\mathrm{e}^t \frac{\mathrm{e}^t}{(1+t^2)}}{\mathrm{e}^{2t}} = \frac{1}{1+t^2}$$

$$u_1 = -\frac{1}{2}\ln(1+t^2) \qquad u_2 = \tan^{-1}(t)$$

Therefore, the full solution is:

$$y_{p_1}(t) = u_1 y_1 + u_2 y_2 = -\frac{1}{2}\mathrm{e}^t \ln(1+t^2) + t\mathrm{e}^t \tan^{-1}(t)$$

Now take $g_2(t) = t^2 \mathrm{e}^t$ and use Method of Undet Coeffs:

$$y_p = (At^2 + Bt + C)\mathrm{e}^t$$

BUT we have to multiply by t^2, so that

$$y_p = t^2(At^2 + Bt + C)\mathrm{e}^t = (At^4 + Bt^3 + Ct^2)\mathrm{e}^t$$

A little messy algebra:

$$\begin{array}{rlllll} +y_p & = e^t(At^4 & +Bt^3 & +Ct^2) & & \\ -2(y_p' & = e^t(At^4 & +(4A+B)t^3 & +(3B+C)t^2 & +2Ct) & \\ y_p'' & = e^t(At^4 & +(8A+B)t^3 & +(12A+6B+C)t^2 & +(6B+4C)t & +2C \\ \hline t^2e^t & = e^t(0 & +0 & +12At^2 & +6Bt & +2C) \end{array}$$

Therefore, $A = 1/12, B = 0, C = 0$, and $y_p = \frac{1}{12}t^4e^t$.

The general solution to $y'' - 2y' + y = \frac{e^t}{1+t^2} + t^2 e^t$ is:

$$y(t) = e^t \left(C_1 + C_2 t + \frac{1}{12} t^4 \right) - \frac{1}{2} e^t \ln(1+t^2) + t e^t \tan^{-1}(t)$$