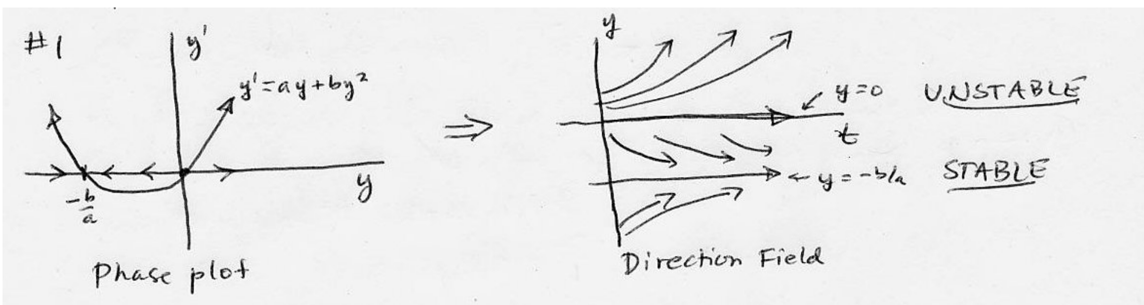
Solutions: Section 2.5

- 2.5, 1: Given $\frac{dy}{dt} = ay + by^2 = y(a + by)$ with $a, b > 0$. For the more general case, we will let y_0 be any real number.

  Always look for the equilibria first! In this case,

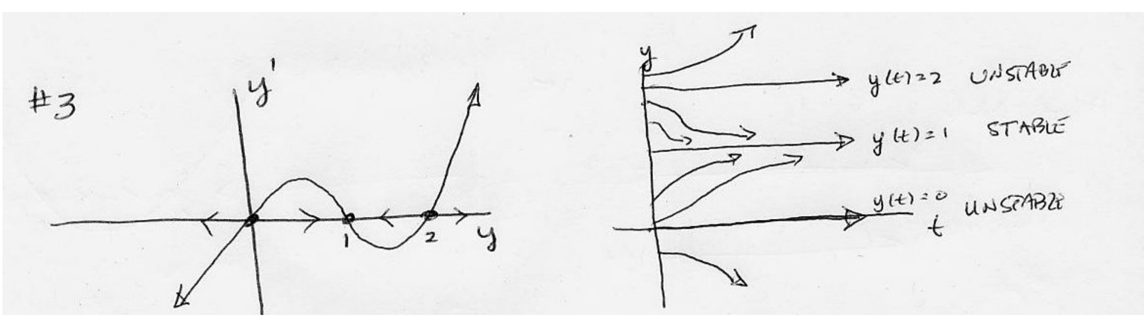
  $$y(a + by) = 0 \quad \Rightarrow \quad y = 0 \text{ or } y = -b/a$$

  To make the phase plot (graph of y' versus y), we note that $ay + by^2$ is a parabola opening upwards, and it intersects the $y-$axis at the equilibria, $y = 0$ and $y = -b/a$. From this graph, we see that $y = 0$ is an unstable equilibrium, and $y = -b/a$ is stable.



- 2.5, 3: Given $\frac{dy}{dt} = y(y - 1)(y - 2)$, and let y_0 be any real number (the more general case).

  Then the phase plot is a cubic function going through the equilibria at $y = 0$, $y = 1$, $y = 2$.

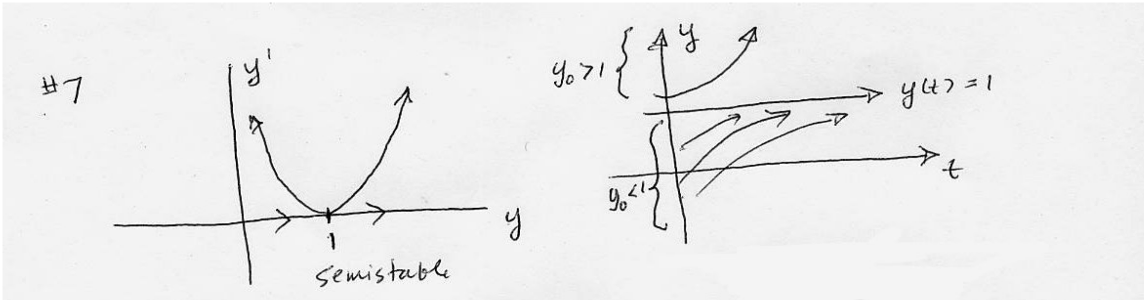


- 2.5, 7: With the DE,

  $$\frac{dy}{dt} = k(1 - y)^2$$

  the only equilibrium solution is: $k(1 - y)^2 = 0 \Rightarrow y = 1$. Graphing this as y' versus y, we get an upward parabola whose vertex is lying on the $y-$axis at $y = 1$.

  For part (b), see the graph.



For part (c), the DE is separable:

$$\int \frac{1}{(1-y)^2}\,dy = \int k\,dt \quad \Rightarrow \quad \frac{1}{1-y} = kt + C$$

(Use u, du substitution for the integral on the left side of the equation). At this stage, we might as well solve for the arbitrary constant:

$$\frac{1}{1-y_0} = 0 + C$$

This is valid as long as $y_0 \neq 1$. In the case that $y_0 = 1$, the solution is $y(t) = 1$ (the equilibrium solution).

Solving for y,

$$1 - y = \frac{1}{kt+C} \quad \Rightarrow \quad y = 1 - \frac{1}{kt + \frac{1}{1-y_0}}$$

Let us analyze this last equation: If $\frac{1}{1-y_0} > 0$, then as $t \to \infty$, $kt + \frac{1}{1-y_0} \to \infty$, so $y(t) \to 1$. Therefore, if $y_0 < 1$, $y(t) \to 1$ as $t \to \infty$ (as expected from the phase plot and direction field).

On the other hand, consider the case when $y_0 > 1$ (the case when $y_0 = 1$ gave an equilibrium solution). In this case, $\frac{1}{1-y_0}$ is negative, which means that there will be a vertical asymptote in positive time (also see figure below)

$$t = -\frac{1}{k(1-y_0)}$$

From our phase plot, we expect solutions with $y_0 > 1$ to go to $+\infty$- Does that occur algebraically?

$$y(t) = 1 - \frac{1}{kt + \frac{1}{1-y_0}} = 1 - \frac{\frac{1}{k}}{t + \frac{1}{k(1-y_0)}}$$

so we see that the denominator is approaching zero *from the left*, so that $y(t) \to +\infty$ as $t \to -1/(k(1-y_0))$ from the left.

- 2.5: 8, 10, 11 are in the Figure below.

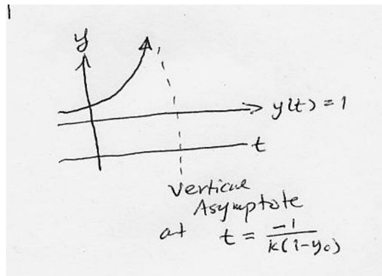
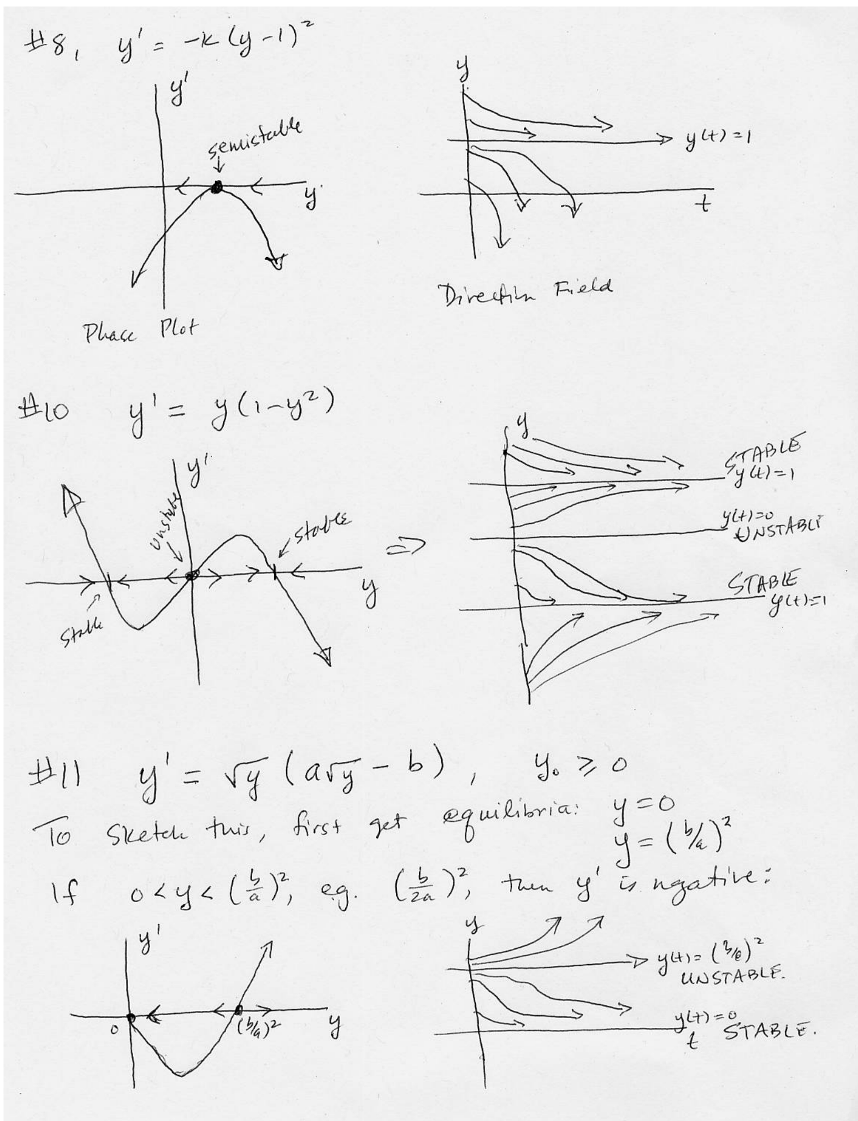


Figure 1: Figure for 7(c) - Note the vertical asymptote.



- Exercise 14: It is OK to argue this graphically, as we did in class. In particular, you should be able to draw a function so that $f(y_0) = 0$ and $f'(y_0) > 0$ (or $f'(y_0) < 0$).

- 2.5, 22: Please be sure to read the description carefully- Nice intro to epidemiology.

  1. The equilibria are at $y = 0$ and $y = 1$. The phase plot of $y' = \alpha y(1 - y)$ is a parabola opening downward. A sketch of the phase plot shows that $y = 0$ is unstable and $y = 1$ is stable.
  2. To solve this, we'll need to use partial fraction decomposition:

$$\frac{1}{y(1-y)} dy = \alpha\, dt \Rightarrow \int \frac{1}{y} + \frac{1}{1-y}\, dy = \alpha t + C \Rightarrow \ln|y| - \ln|1-y| = \alpha t + C$$

so that

$$\ln\left|\frac{y}{1-y}\right| = \alpha t + C \quad \Rightarrow \quad \frac{y}{1-y} = A\mathrm{e}^{\alpha t}$$

Solving for A, $y_0/(1 - y_0) = A$. Keep this in mind, and let's solve for y first:

$$y(t) = \frac{A\mathrm{e}^{\alpha t}}{1 + A\mathrm{e}^{\alpha t}}$$

We will want to analyze what happens as $t \to \infty$, so it will be more convenient to divide numerator and denominator by $A\mathrm{e}^{\alpha t}$:

$$y(t) = \frac{1}{\frac{1}{A}\mathrm{e}^{-\alpha t} + 1} = \frac{1}{\frac{1-y_0}{y_0}\mathrm{e}^{-\alpha t} + 1}$$

This solution is valid as long as $y_0 \neq 0$ and $y_0 \neq 1$. In those cases, our solutions are the equilibrium solutions, $y(t) = 0$ and $y(t) = 1$. Now let us analyze the behavior of $y(t)$.

We see that, as $t \to \infty$, $y(t) \to 1$. But this is not the end of the story: If a solution begins with $y_0 < 0$, for example, we know that the solution CANNOT approach 1 as $t \to \infty$, because that would mean it would have to cross $y(t) = 0$ (and solutions cannot intersect by the E& U Theorem).

The following is a much more detailed analysis than what was expected in the homework problem- However, read through it to see exactly what the behavior of all solutions looks like.

The only point that makes us pause is the denominator. Set it to zero and solve:

$$\frac{1-y_0}{y_0}\mathrm{e}^{-\alpha t} = -1 \quad \Rightarrow \quad \mathrm{e}^{-\alpha t} = \frac{y_0}{y_0 - 1} \quad \Rightarrow \quad t = -\frac{1}{\alpha} \cdot \ln\left(\frac{y_0}{y_0 - 1}\right)$$

Alternatively,

$$t = \frac{1}{\alpha} \cdot \ln\left(\frac{y_0 - 1}{y_0}\right) = \frac{1}{\alpha} \cdot \ln\left(1 - \frac{1}{y_0}\right)$$

The reason this is a nice way of analyzing t:

- If $y_0 > 1$, then we will be taking the log of a number less than 1 (which gives a negative value). In this case, t is negative and our solution $y(t)$ is valid for all $t > (1/\alpha)\ln(1-(1/y_0))$, and $y(t) \to 1$ as $t \to \infty$.
- If $0 < y_0 < 1$, this denominator is never zero (no solution for t in the real numbers). In this case, $y(t)$ is valid for ALL t (not just positive), and again the limit as $t \to \infty$ is 1.
- If $y_0 < 0$, then the solution is valid for:

$$-\infty < t < \frac{1}{\alpha}\ln\left(1 - \frac{1}{y_0}\right)$$

so that $y(t)$ has a vertical asymptote on the positive t axis. In this case, it is not appropriate to take the limit as $t \to \infty$.

- 2.5, 23:

First solve $y' = -\beta y$, which is $y(t) = y_0 \mathrm{e}^{-\beta t}$.

NOTE: There is a misprint in Problem 23, in defining dx/dt. The disease *spreads* (or INCREASES) at a rate proportional to the number of carrier-susceptible interactions ($x-$ and $y-$ interactions), which means that the constant in front should be POSITIVE.

We are told to substitute this into the DE:

$$\frac{dx}{dt} = +\alpha x y = \alpha x\left(y_0 \mathrm{e}^{-\beta t}\right)$$

Solve this separable equation for $x(t)$:

$$\int \frac{1}{x}\,dx = \alpha y_0 \int \mathrm{e}^{-\beta t}\,dt \quad \Rightarrow \quad \ln|x| = \frac{-\alpha \cdot y_0}{\beta}\mathrm{e}^{-\beta t} + C$$

Solving for the initial value,

$$C = \ln|x_0| + \frac{\alpha \cdot y_0}{\beta}$$

so that:

$$\ln|x| = \frac{\alpha \cdot y_0}{\beta}\left(1 - \mathrm{e}^{-\beta t}\right) + \ln|x_0|$$

Finally, exponentiating both sides:

$$x(t) = x_0 \mathrm{e}^{\frac{\alpha \cdot y_0}{\beta}\left(1-\mathrm{e}^{-\beta t}\right)}$$

And the limit as $t \to \infty$ of $x(t)$ is $x_0 \mathrm{e}^{\frac{\alpha \cdot y_0}{\beta}}$

- 2.5, 24

  I hope you're asking yourself what it is we're doing in this problem:

  The text is getting to a "normalized" model of the disease, where at time 0 none of the population has the disease ($z(0) = 1.00$ or $z(0) = 100\%$), then as time goes on, we're modeling the percentage of the population that has not yet been exposed to smallpox- That is,

  $$z(t) = \frac{\text{Number of people who have not been exposed to smallpox at time } t}{\text{Number of people who are (still) alive at time } t} = \frac{x(t)}{n(t)}$$

  This is an interesting way of doing the modeling, since we are focused on a single "cohort".

  For our model, the susceptible population will only decline either due to *exposure to* smallpox (β is the exposure rate, ν is the death rate. *Side Remark: The greek symbol ν is read as "nu", or "noo"*) or death from something else:

  $$\frac{dx}{dt} = -(\text{Exposure rate-Smallpox}) - (\text{Death rate from other})$$

  The constants are typically given as proportions- That is, the overall exposure rate to smallpox would be $\beta x(t)$, and we're told that the death rate will be $\mu(t)x(t)$. Putting these together gives us the text's equation:

  $$\frac{dx}{dt} = -(\beta + \mu)x$$

  Now, we might notice that since ν is the death rate (as a proportion) from smallpox, and β is the exposure rate, then the overall death rate in the population due to smallpox will be $\nu\,\beta\,x(t)$. Similarly, we need to take away the population that has died from other causes, $\mu(t)n(t)$ (recall that $n(t)$ is the number of people alive at time t). Now we have the DE for $n(t)$:

  $$\frac{dn}{dt} = -\nu\beta x(t) - \mu(t)n(t) = -\nu\beta x - \mu n$$

  Now we get to the questions:

  (a) Let $z = x/n$. Then

  $$\frac{dz}{dt} = \frac{x'\,n - x\,n'}{n^2} = \frac{-\beta xn - \mu xn - x(-\nu\beta x - \mu n)}{n^2} = -\beta z(1 - \nu z)$$

  And, since $x(0) = n(0)$ (everyone is alive and susceptible at time 0), then $z(0) = 1$ (or 100%).

(b) To solve the DE, we see that

$$\int \frac{1}{z(1-\nu z)}\,dz = \int -\beta\,dt$$

So that, using partial fractions on the left, we get

$$z(t) = \frac{1}{(1-\nu)e^{\beta t} + \nu}$$

Using the suggested values of $\nu = \beta = 1/8$ and $t = 20$, we get $z \approx 0.093$, so after 20 years, only about 9.2% of the population remain unexposed to smallpox.

- 2.5, 25: The basic idea behind problems 25 and 26 is that there is a new parameter, a. By changing this parameter, we can change the *number* and *type* of the equilibrium solutions.

  In Problem 25, the equilibrium solutions are given by:

  $$\frac{dy}{dt} = 0 \Rightarrow a - y^2 = 0 \quad \Rightarrow \quad y = \pm\sqrt{a}$$

  Graphically in the phase plot, $y' = -y^2$ is an upside down parabola, and $-y^2 + a$ simply translates the parabola up and down.

  Therefore, in words:

  - If $a < 0$, we have no equilibrium solutions.
  - If $a = 0$, we have a single equilibrium solution at $a = 0$, and it is *semistable*. Since y' is always negative (and zero at $y = 0$), in the direction field, solutions that begin above $y_0 = 0$ decrease to zero, and solutions that begin below $y_0 = 0$ decrease to negative infinity.
  - If $a > 0$, we have two equilibrium solutions (at $\sqrt{a}$ and $-\sqrt{a}$). The positive root is a *stable* equilibrium, and the negative root is an *unstable* equilibrium.

  We can summarize this graphically in Figure 2.5.10 on page 93.

- Problem 26: Finding the equilibrium:

  $$y(a - y^2) = 0$$

  We see that $y(t) = 0$ is ALWAYS an equilibrium solution for any value of a. The other solutions will be the same as before (we'll have to re-do our stability analysis):

  - If $a < 0$, the only equilibrium is $y(t) = 0$, and this is stable.
  - If $a = 0$, same situation.
  - If $a > 0$, two new equilibria appear, $y(t) = \pm\sqrt{a}$. Now, $y(t) = 0$ switches stability (it is now unstable), and the two new equilibria, $y(t) = \pm\sqrt{a}$ are both stable.