Selected solutions, 2.6

NOTE: In Problems 1 and 3, we show two different ways of finding the underlying function f. Either way is fine.

1. Take $M = 2x + 3$ and $N = 2y - 2$. Then $M_y = N_x = 0$, and to find the solution, we can antidifferentiate M:
$$f(x, y) = \int M\,dx = x^2 + 3x + h_1(y)$$
We can differentiate this to see if we get N: $f_y = h_1'(y) = 2y - 2$. Therefore, $h_1(y) = y^2 - 2y$. Put it all together to get the solution:
$$x^2 + 3x + y^2 - 2y = C$$

3. $(3x^2 - 2xy + 2)\,dx + (6y^2 - x^2 + 3)\,dy = 0$

SOLUTION: If this is exact, this is of the form $f_x\,dx + f_y\,dy$ for some f. We use the test ($M_y = N_x$):
$$M_y = -2x \qquad N_x = -2x$$
Now we try to reconstruct f. One way to do it is to integrate twice and compare:
$$f_x = 3x^2 - 2xy + 2 \quad \Rightarrow \quad f(x, y) = x^3 - x^2y + 2x + h_1(y)$$
Using the other function,
$$f_y = 6y^2 - x^2 + 3 \quad \Rightarrow \quad f(x, y) = 2y^3 - x^2y + 3y + h_2(x)$$
Now compare the two expressions to see that $f(x, y) = x^3 + 2y^3 - x^2y + 2x + 3y$ and the general solution is:
$$x^3 + 2y^3 - x^2y + 2x + 3y = C$$

4. Solving it as written, we should get $x^2y^2 + 2xy = C$, but did you notice that you could factor $2xy + 2$ out of M and N (if not, that's OK). In that case, the equation becomes simpler- In fact, we end up $xy = C$ as the solution.

10. (We started it in class)
$$(y/x + 6x)\,dx + (ln(x) - 2)\,dy = 0 \qquad x > 0$$
Checking, we see that $M_y = 1/x = N_x$, so the DE is exact. Integrating M first gives:
$$f(x, y) = y\ln(x) + 3x^2 + G_1(y)$$
Integrating N gives:
$$f(x, y) = y\ln(x) - 2y + G_2(x)$$
Comparing these, we see that $f(x, y) = y\ln(x) + 3x^2 - 2y$ (NOTE: We are NOT adding these together; we are comparing them). The overall general solution is then
$$y\ln(x) + 3x^2 - 2y = C$$

13. You should get that the general solution is

$$x^2 - xy + y^2 = C$$

and that the initial condition yields $c = 7$. In this case, one could solve the specific solution for y by completing the square in y, or you could use the quadratic formula in y. Algebraically, this gets a little messy, but here it is:

$$y^2 - xy = 7 - x^2 \quad \Rightarrow \quad y^2 - xy + \frac{x^2}{4} = 7 - \frac{3x^2}{4} =$$

$$\left(y - \frac{x}{2}\right)^2 = \frac{28 - 3x^2}{4} \quad \Rightarrow \quad y = \frac{x + \sqrt{28 - 3x^2}}{2}$$

(Positive root to match the IC) so the solution is valid as long as $3x^2 \leq 28$.

17. There is a small (but important) typo. The function ψ (the symbol ψ is read like the beginning of "psychology", psi):

$$\psi(x, y) = \int_{x_0}^{x} M(s, y_0)\, ds + \int_{y_0}^{y} N(x_0, t)\, dt$$

Having said that, I believe this question is suggesting something that is incorrect. If you have the time and interest, you might try working out a specific example to see what happens.

18. In the case that M is a function of x alone, and N is a function of y alone, then $M_y = N_x = 0$.

19. Multiply by the integrating factor so that the new DE is exact:

$$\frac{x^2y^3}{xy^3} + \frac{x(1+y^2)}{xy^3} y' = 0 \quad \Rightarrow \quad x + \frac{1+y^2}{y^3} y' = 0$$

This is like Exercise 18- This is a separable DE, but we'll solve it as an exact equation:

$$M(x, y) = x \quad \Rightarrow \quad f(x, y) = \frac{1}{2}x^2 + h_1(y)$$

$$N(x, y) = \frac{1}{y^3} + \frac{1}{y} \quad \Rightarrow \quad f(x, y) = -\frac{1}{2y^2} + \ln|y| + h_2(x)$$

Put these together:

$$\frac{1}{2}x^2 - \frac{1}{2y^2} + \ln|y| = C$$

(The text multiplied everything by 2)

22. With the given integrating factor, we write:

$$(x + 2)xe^x \sin(y) + x^2 e^x \cos(y)\, y' = 0$$

Now this is exact, since $M_y = (x^2 + 2x)e^x \cos(y)$, and so is N_x (remember to use the product rule). Now to integrate, the author has been kind to us- Remember, we can choose which integral to do- Either M with respect to x (a little messy) or N with respect to y (easy!):

$$f(x, y) = \int N\, dy = x^2 \mathrm{e}^x \sin(y) + h_1(x)$$

and now determine if there is a function $h_1(x)$ by comparing f_x to M:

$$f_x = \sin(y) \left(2x\mathrm{e}^x + x^2 \mathrm{e}^x\right) + h_1'(x) \quad \Rightarrow \quad h_1(x) = 0$$

and the solution is:

$$x^2 \mathrm{e}^x \sin(y) = C$$