Selected Solutions, Section 3.2

13. This is a specific illustration of the superposition principle. Note that $t^2y'' - 2y = 0$ is a linear, second order, homogeneous DE, and we are given $y_1 = t^2$ and $y_2 = t^{-1}$. We then show (by substitution) that anything of the form $c_1y_1 + c_2y_2$ is also a solution.

14. In contrast to 13, we are given two solutions to a nonlinear DE, and we show that superposition does not work. You should first verify that each function is itself a solution. Then

$$y = c_1 + c_2 t^{1/2} \qquad y' = \frac{c_2}{2}t^{-1/2} \qquad y'' = \frac{-c_2}{4}t^{-3/2}$$

so that

$$y\,y'' = -\frac{c_1c_2}{4}t^{-3/2} - \frac{c_2^2}{4}t^{-1}$$

while

$$(y')^2 = \frac{c_2^2}{4}t^{-1}$$

and adding these clearly will not be zero unless $c_1 = 0$.

15. Shows the importance of superposition to only HOMOGENEOUS equations. To write the solution using operator notation, we can take

$$L(y) = y'' + p(t)y' + q(t)y$$

where L is a linear operator, and we're told that $L(\phi) = g(t)$. In that case,

$$L(c\phi) = cL(\phi) = cg(t) \neq g(t)$$

unless $c = 1$. Therefore, if ϕ solves the *nonhomogeneous* DE, we cannot take $c\phi$ as a more general solution (as we did for homogeneous equations).

16. In this case, go ahead and substitute the given expression into the differential equation. We will be able to find expressions for $p(t)$ and $q(t)$:

$$y = \sin(t^2) \qquad y' = 2t\cos(t^2) \qquad 2\cos(t^2) - 4t^2\sin(t^2)$$

so that $y'' + p(t)y' + q(t)y = 0$ becomes the following, after collecting terms with $\cos(t^2)$ together, and $\sin(t^2)$ together:

$$(2 + 2tp(t))\cos(t^2) + (-4t^2 + q(t))\sin(t^2) = 0$$

Since this must be zero for all t, each coefficient must be zero:

$$q(t) = 4t^2$$

and

$$2 + 2p(t) = 0 \quad \Rightarrow \quad p(t) = \frac{1}{t}$$

which is not continuous at $t = 0$.

17. Illustrates the important technique we also discussed in class- Given the Wronskian (either as an expression, as in this exercise, or via Abel's Theorem as in class), and one solution f, we can find a second solution to the linear second order DE. In this case, we find that

$$W(f,g) = e^{2t}g' - 2e^{2t}g = 3e^{4t}$$

which is a first order DE for $g(t)$. Solving, we get:

$$g' - 2g = 3e^{2t} \qquad (ge^{-2t})' = 3$$

and continuing, the general function g is:

$$g(t) = 3te^{2t} + Ce^{2t}$$

Since $f = e^{2t}$, we can set $C = 0$ and just use $3te^{2t}$.

18. Same idea as 17.

19. Exercise 21 generalizes Exercise 19 and 20, so you can either do the generalization first, or simply multiply out the expressions and see if you can find the Wronskian. Here is the way it works in general:

Given

$$y_3 = a_1y_1 + a_2y_2 \qquad y_4 = b_1y_1 + b_2y_2$$

Let's see if we can find an expression for $W(y_3, y_4)$ in terms of $W(y_1, y_2)$.

By definition, $W(y_3, y_4)$ can be written as:

$$\begin{vmatrix} a_1y_1 + a_2y_2 & b_1y_1 + b_2y_2 \\ a_1y_1' + a_2y_2' & b_1y_1' + b_2y_2' \end{vmatrix} = (a_1y_1 + a_2y_2)(b_1y_1' + b_2y_2') - (a_1y_1' + a_2y_2')(b_1y_1 + b_2y_2)$$

which is multiplied out and simplified:

$$\begin{array}{cccc} a_1b_1y_1y_1' & +a_1b_2y_1y_2' & +a_2b_1y_1'y_2 & +a_2b_2y_2y_2'+ \\ -a_1b_1y_1y_1' & -a_2b_1y_1y_2' & -a_1b_2y_1'y_2 & -a_2b_2y_2y_2' \\ \hline & (a_1b_2 - a_2b_1)y_1y_2' & -(a_1b_2 - a_2b_1)y_1'y_2 & \end{array}$$

From which we get our result:

$$(a_1b_2 - a_2b_1)(y_1y_2' - y_1'y_2) = (a_1b_2 - a_2b_1)W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

For people who have had linear algebra, you might recognize this as:

$$\det(AB) = \det(A)\det(B)$$

where

$$A = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \qquad B = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$

NOTE: If you have not had linear algebra, it is safe to ignore this, and just multiply things out...

For exercise 19, if $u = 2f - g$ and $v = f + 2g$, then

$$W(u, v) = \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} W(f, g) = 5W(f, g)$$

22. Good for practice, lots of algebra. Using Wolfram Alpha, we can write y_1 as the solution to the following:

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solve y''+y'-2y=0, y(0)=1, y'(0)=0
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which gives $y_1 = \frac{1}{3}\mathrm{e}^{-2t} + \frac{2}{3}\mathrm{e}^{t}$ and y_2 is the solution to the following:

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solve y''+y'-2y=0, y(0)=0, y'(0)=1
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which is $y_2 = -\frac{1}{3}\mathrm{e}^{-2t} + \frac{1}{3}\mathrm{e}^{t}$

We can contrast this with the usual method for finding a fundamental set, $y_1 = \mathrm{e}^{-2t}$ and $y_1 = \mathrm{e}^{t}$ (that's why in class we said that Theorem 3.2.5 is more of a theoretical result rather than a computational one).

26. Straightforward- Verify that each function is indeed a solution, then compute the Wronskian.

27. In this case, the discontinuity is a little tricky. Notice that

$$p(x) = -\frac{x}{1 - x\cot(x)}$$

so that points of discontinuity are where $1 - x\cot(x) = 0$. Simplifying this a bit, we get

$$1 - \frac{x\cos(x)}{\sin(x)} = 0 \quad \Rightarrow \quad x\cos(x) = \sin(x) \quad \Rightarrow \quad x\cos(x) - \sin(x) = 0$$

And this just happens to be where the Wronskian is zero.

29. Use Abel's Theorem to compute the Wronskian (up to a constant multiple):

$$W = C\mathrm{e}^{\int p(t)\,dt} = C\mathrm{e}^{\int 1+(2/t)\,dt} = Ct^2\mathrm{e}^{t}$$

31. Same idea as 29.

34. The extra information, $W(y_1, y_2)(1) = 2$ allows us to solve for the constant C from Abel's Theorem. Therefore,

$$W(y_1, y_2)(t) = \frac{C}{t^2}$$

and using the extra info, $C = 2$. Therefore,

$$W(y_1, y_2)(5) = \frac{2}{5^2} = \frac{2}{25}$$

36. If $Ce^{-\int p(t)\,dt}$ is constant, we must have $p(t) = 0$.

38. If y_1, y_2 are both zero at some point (call it t^*), then the Wronskian is zero at t^*:

$$W(y_1, y_2)(t^*) = \begin{vmatrix} y_1(t^*) & y_2(t^*) \\ y_1'(t^*) & y_2(t^*) \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ y_1'(t^*) & y_2(t^*) \end{vmatrix} = 0$$