Selected Solutions, Section 3.3 (Complex)

7. Solving the characteristic equation, $r = 1 \pm i$. Therefore,

$$y = e^t \left(C_1 \cos(t) + C_2 \sin(t)\right)$$

8. Solving the characteristic equation, $r = 1 \pm \sqrt{5}\, i$. Therefore,

$$y = e^t \left(C_1 \cos(\sqrt{5}\, t) + C_2 \sin(\sqrt{5}\, t)\right)$$

9. Solving the characteristic equation, $r = 2, -4$. Therefore,

$$y = C_1 e^{2t} + C_2 e^{-4t}$$

(Similarly, we solve 10-15).

25. For this one, try to go as far as you can without assistance from the computer: Let $y'' + 2y' + 6y = 0$ with $y(0) = 2$ and $y'(0) = \alpha \geq 0$.

(a) Solve it: The characteristic equation is $r^2 + 2r + 6 = 0$. Therefore, $r = -1 \pm \sqrt{5}\, i$ and the general solution is:

$$y(t) = e^{-t} \left(C_1 \cos(\sqrt{5}\, t) + C_2 \sin(\sqrt{5}\, t)\right)$$

With the initial conditions, find that $C_1 = 2$ and $C_2 = \frac{\alpha+2}{\sqrt{5}}$, so that the solution to the IVP is:

$$y(t) = e^{-t} \left(2\cos(\sqrt{5}\, t) + \frac{\alpha + 2}{\sqrt{5}} \sin(\sqrt{5}\, t)\right)$$

(b) Find α so that $y(1) = 0$: Algebraically, we get:

$$-2\sqrt{5}\, \frac{\cos(\sqrt{5})}{\sin(\sqrt{5})} - 2 = \alpha$$

(c) Find the smallest value of $t > 0$ so that $y(t) = 0$. We'll write our answer as a function of α. We are meant to use technology on this problem, but we could do it by hand as follows:

Algebraically, solving this:

$$0 = e^{-t} \left(2\cos(\sqrt{5}\, t) + \frac{\alpha + 2}{\sqrt{5}} \sin(\sqrt{5}\, t)\right)$$

will get us to:

$$\frac{-2\cos(\sqrt{5}\, t)}{\sin(\sqrt{5}\, t)} = \frac{\alpha + 2}{\sqrt{5}}$$

We want to solve this for t, remembering that we want the smallest $t > 0$. I think it is easiest to work with the tangent rather than the cotangent,

$$\tan(\sqrt{5}\, t) = \frac{-2\sqrt{5}}{\alpha + 2}$$

Therefore:

$$t = \frac{1}{\sqrt{5}} \tan^{-1}\left(\frac{-2\sqrt{5}}{\alpha+2}\right)$$

And, when $\alpha \to \infty$, $t \to 0$.

27. Straightforward computation. Recall what we said in class- If we let $r = \lambda + i\mu$, then

$$y_1 = \mathrm{Re}(\mathrm{e}^{rt}) = \mathrm{Re}(\mathrm{e}^{\lambda t+(\mu t)i}) = \mathrm{e}^{\lambda t}\cos(\mu t)$$

and

$$y_2 = \mathrm{Im}(\mathrm{e}^{rt}) = \mathrm{Im}(\mathrm{e}^{\lambda t+(\mu t)i}) = \mathrm{e}^{\lambda t}\sin(\mu t)$$

then $W(y_1, y_2) = \mu \mathrm{e}^{2\lambda t} \neq 0$, so y_1, y_2 will form a fundamental set of solutions to our second order linear homogeneous DE with constant coefficients (in the case where we have complex roots).

35, 37 **Euler Equations** (We're actually doing these after Section 3.4 this time, so I'm repeating this stuff there, too).

The first thing is to convert y from a function of t to a function of x using $x = \ln(t)$. The text tell us to write dy/dt and d^2y/dt^2 in terms of dy/dx and d^2y/dx^2. To do that, we use the Chain Rule. Doing this using Leibniz' notation is easiest:

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

From $x = \ln(t)$, we have $dx/dt = 1/t$, so that

$$\frac{dy}{dx} = \frac{1}{t}\frac{dy}{dx}$$

Secondly, we find a formula for d^2y/dt^2. To do that, we use our previous answer and the Product Rule:

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{1}{t}\frac{dy}{dx}\right) = -\frac{1}{t^2}\frac{dy}{dx} + \frac{1}{t}\frac{d}{dt}\left(\frac{dy}{dx}\right) =$$

In this second term, use the chain rule again. The form we are using is the same as before:

$$\frac{d}{dt}(\cdot) = \frac{d\cdot}{dx}\frac{dx}{dt}$$

Where the first time around, the dot was y, the second time the dot is dy/dx. Therefore,

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}\frac{dx}{dt} = \frac{1}{t}\frac{d^2y}{dx^2}$$

Putting it all together, with prime denoting the derivative in x, we have:

$$\frac{d^2y}{dt^2} = \frac{y'' - y'}{t^2}$$

Finally, we substitute into the given equation in t to get a DE in x:

$$t^2\frac{d^2y}{dt^2} + \alpha\frac{dy}{dt} + \beta y = 0 \tag{1}$$

Substitute:

$$t^2\left(\frac{y'' - y'}{t^2}\right) + \alpha t\left(\frac{1}{t}\frac{dy}{dt}\right) + \beta y = 0 \quad \Rightarrow \quad y'' + (\alpha - 1)y' + \beta y = 0 \tag{2}$$

Now, to solve Equation 1, we solve 2 and use the substitution $x = \ln(t)$.

As a side remark, you might notice the following: If we use the ansatz $y = t^r$ for Equation 1, we get the following:

$$t^2(r(r-1)t^{r-2}) + \alpha t\, rt^{r-1} + \beta t^r = 0 \quad \Rightarrow \quad r^2 + (\alpha - 1)r + \beta = 0$$

which is the same characteristic equation that is associated to Equation 2. We then have three cases (as for the case in Sections 3.1-3.4).

Important Summary To solve

$$t^2\frac{d^2y}{dt^2} + \alpha t\frac{dy}{dt} + \beta y = 0,$$

we use the ansatz $y = t^r$, from which we get the new characteristic equation:

$$r(r-1) + \alpha r + \beta = 0 \quad \text{or} \quad r^2 + (\alpha - 1)r + \beta = 0$$

In solving the characteristic equation, we have three cases:

- Two real solutions: r_1, r_2. We can write the solution in x, then translate to t using $x = \ln(t)$:

$$y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x} \quad \Rightarrow \quad y(t) = C_1 t^{r_1} + C_2 t^{r_2}$$

- One real solution, r. Using the same technique, first write in x, then translate to t:

$$y(x) = e^{rx}(C_1 + C_2 x) \quad \Rightarrow \quad y(t) = t^r(C_1 + C_2\ln(t))$$

- Complex solutions, $r = \lambda + \gamma i$ (we only need one of them). The solution is:

$$y = e^{\alpha x}(C_1\cos(\beta x) + C_2\sin(\beta x)) \quad \Rightarrow \quad y = t^\alpha\left(C_1\cos(\ln(t^\beta) + C_2\sin(\ln(t^\beta))\right)$$

37. Solve the following, using Equation 2.

$$t^2\ddot{y} + 3t\dot{y} + \frac{5}{4}y = 0$$

The characteristic equation is:

$$r(r-1) + 3r + \frac{5}{4} = 0 \quad \Rightarrow \quad r^2 + 2r + \frac{5}{4} = 0 \quad \Rightarrow \quad r = -1 \pm \frac{1}{2}i$$

so that, in x we would have the following, and then we translate to t:

$$y(x) = e^{-x}\left(C_1\cos\left(x/2\right) + C_2\sin\left(x/2\right)\right)$$

$$y(t) = \frac{1}{t}\left(C_1\cos(\ln(\sqrt{t}) + C_2\sin(\sqrt{t})\right)$$