Solutions to Selected Problems, 3.7 (Model of Mass-Spring System)

NOTE about units: On quizzes/exams, we will always use the standard units of meters, kilograms and seconds, or feet, pounds and seconds. The textbook likes to mix them up somewhat.
3.7, 1, 2, 5-7, 11, 14. Also: Sect 3.4, 38 and 39

1-2 The first two exercises are for practice using Equation 17 (p 195)

5. A mass weighing 2 lb stretches a spring 6 inches.

   Remark: This information is here so that we can get the spring constant. Change the 6 inches to 1/2 foot:

$$mg - kL = 0 \quad \Rightarrow 2 - \frac{k}{2} = 0 \quad \Rightarrow \quad k = 4$$

   From this, we can also get the mass using $g = 32$ ft/sec^2 (the constant would be given to you):

$$mg = 2 \quad \Rightarrow \quad m = \frac{2}{g} = \frac{2}{32} = \frac{1}{16}$$

   Continuing with the problem, we only need to determine γ- Since there is no damping, $\gamma = 0$, and

$$\frac{1}{16}u'' + 0u' + 4u = 0 \quad \Rightarrow \quad u'' + 64u = 0$$

   If the mass is pulled down 3 inches and released, the initial conditions are $u(0) = \frac{1}{4}$ and $u'(0) = 0$. Solving the IVP, we get

$$u(t) = \frac{1}{4}\cos(8t)$$

   so the amplitude is 1/4 and the period is $2\pi/8$ or $\pi/4$. You don't need to plot it for now.

6. First thing is that your units should stay kg, meters, seconds.

   So, the mass is 100 g = 0.1 kg and the length is 5 cm = 0.05 m (I will give you "mks" units on the exam/quiz). Given these values, it is straightforward to proceed. First the spring constant:

$$mg - kL = 0 \quad \Rightarrow \quad (0.1)(9.8) - k(0.05) = 0 \quad \Rightarrow \quad k = 19.6$$

   That gives the DE:

$$\frac{1}{10}u'' + 19.6u = 0 \qquad u(0) = 0, \quad u'(0) = \frac{1}{10}$$

(NOTE: We said that the force for the weight was mg, not $-mg$, so we are actually setting "down" as positive $u(t)$).

Multiply by 10 before solving:

$$u'' + 196u = 0 \quad \Rightarrow \quad u(t) = C_1\cos(14\,t) + C_2\sin(14\,t)$$

Solving for the coefficients, we should get:

$$u(t) = \frac{1}{140}\sin(14\,t)$$

When does the mass return to equilibrium? We know that $\sin(A) = 0$ when $A = 0$ and $A = \pi$, so in this case, we take

$$14t = \pi \qquad t = \frac{\pi}{14} \approx 0.224 \text{ rad}$$

7. In this case, we have US Customary units, so use $g = 32\text{ft/sec}^2$, and use lbs, feet and seconds.

For the spring constant: $mg - kL = 0$ means that

$$3 - \frac{k}{4} = 0 \quad \Rightarrow \quad k = 12$$

We have no damping, $u(0) = -\frac{1}{12}\text{ft}$ (the spring is compressed), and $u'(0) = 2\text{ft/sec}$.

Therefore,

$$\frac{3}{32}u'' + 12u = 0 \qquad u(0) = -\frac{1}{12} \quad u'(0) = 2$$

We might notice that this does simplify a bit by multiplying through by 32/3:

$$u'' + 128u = 0 \qquad u(0) = -\frac{1}{12} \quad u'(0) = 2$$

(Note that $128 = 64 \times 2$)

$$u(t) = C_1\cos(8\sqrt{2}\,t) + C_2\sin(8\sqrt{2}\,t)$$

The initial conditions give us the equations:

$$C_1 = -\frac{1}{12} \qquad C_2 = \frac{1}{4\sqrt{2}}$$

Writing the solution as $R\cos(\omega t - \delta)$, we find that:

$$R = \sqrt{C_1^2 + C_2^2} = \sqrt{\frac{1}{144} + \frac{1}{32}} = \sqrt{\frac{11}{288}}$$

$$\omega = 8\sqrt{2} \qquad \delta = \tan^{-1}\left(\frac{3}{\sqrt{2}}\right)$$

The phase shift is δ, the amplitude is given by R. The frequency F, period P, are given by:

$$F = \frac{4\sqrt{2}}{\pi} \qquad P = \frac{2\pi}{8\sqrt{2}} = \frac{\pi}{4\sqrt{2}}$$

11. (Watch the units!) Building the model, the spring constant is

$$k = \frac{3}{0.1} = 30 \text{ N/m}$$

and the damping coefficient:

$$\gamma u' = F_d \quad \Leftrightarrow \gamma(5) = 3 \quad \Rightarrow \gamma = \frac{3}{5}$$

so that, keeping meters as the unit of length (and downward as positive):

$$2u'' + \frac{3}{5}u' + 30u = 0 \qquad u(0) = \frac{5}{100} = \frac{1}{20} \quad u'(0) = \frac{1}{10}$$

Divide by 2 if you want to complete the square:

$$u'' + \frac{3}{10}u' + 15u = 0 \quad \Rightarrow \quad r^2 + \frac{3}{10}r + \frac{3^2}{20^2} = -15 + \frac{9}{20^2} = \frac{9-6000}{20^2} = \frac{5991}{20^2}$$

Using exact arithmetic, we have the following, which we can approximate:

$$r = -\frac{3}{20} \pm \frac{\sqrt{5991}}{20}\,i \qquad\qquad r \approx -0.15 \pm 3.87\,i$$

Therefore, if we let $\mu = 3.87$, we have:

$$u(t) = \mathrm{e}^{-0.15t}(C_1\cos(\mu t) + C_2\sin(\mu t))$$

so that we have the two equations in C_1, C_2:

$$\begin{aligned} C_1 &= 1/20 = 0.05 \\ -0.15C_1 + \mu C_2 &= 1/10 = 0.1 \end{aligned} \quad \Rightarrow \quad C_1 = 0.05 \quad C_2 \approx 0.0045.$$

14. The period of motion of an undamped (unforced) spring is the period of the homogeneous part of the solution to $mu'' + ku = 0$, which is

$$\frac{2\pi}{\sqrt{k/m}} = 2\pi\sqrt{\frac{m}{k}}$$

At equilibrium, $mg = kL$, so that $\frac{m}{k} = \frac{L}{g}$, so we can write the equation above as:

$$2\pi\sqrt{\frac{L}{g}}$$

Section 3.4 Exercises

These are only about analyzing our possible solutions to:

$$mu'' + \gamma u' + ku = 0$$

if we force $m > 0$ and $\gamma, k \geq 0$.

SOLUTION: Since m, γ, k are all positive, then

- Case 1: If $\gamma^2 - 4mk > 0$, then $\gamma^2 - 4mk < b^2$, so the roots are both negative:

  $$r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m} < 0$$

  Lets call the roots $-r_1, -r_2$ just to emphasize the fact that they are negative. Then

  $$u(t) = C_1 \mathrm{e}^{-r_1 t} + C_2 \mathrm{e}^{-r_2 t}$$

  which goes to zero as $t \to \infty$, for any choice of C_1, C_2.

- Case 2: If $\gamma^2 - 4mk < 0$, then the real part of the complex root r is $-\gamma/2m$, which again is negative (also see below)

  $$r = \frac{-\gamma}{2m} \pm \frac{\sqrt{4mk - \gamma^2}}{2m} i = \frac{-\gamma}{2m} \pm \beta i$$

  The general solution is then:

  $$u(t) = \mathrm{e}^{-(\gamma/2m)t} \left(C_1 \cos(\beta t) + C_2 \sin(\beta t)\right)$$

  The exponential function will force the rest of the solution to zero as t goes to infinity.

  SPECIAL CASE: $\gamma = 0$. With no damping, then $r = \pm \beta i$ (same β as before), so that

  $$u(t) = C_1 \cos(\beta t) + C_2 \sin(\beta t)$$

  which does NOT go to zero, but remains bounded (and is periodic).

- Case 3: $\gamma^2 - 4mk = 0$. In this case, $r = -\gamma/2m$ (which is negative), and the general solution is

  $$u(t) = \mathrm{e}^{-(\gamma/2m)t}(C_1 + C_2 t)$$

  which again goes to zero as $t \to \infty$.

Therefore, in all cases but one (where damping is zero), the general solution to the homogeneous equation always tends to zero as $t \to \infty$.