Selected Solutions to 3.8

In Exercises 1-2, use the formulas given in the text to re-write the function as a product. You do not need to memorize them.

5. The mass is $4/32 = 1/8$. We find the spring constant as usual:

$$mg - kL = 0 \quad \Rightarrow \quad 4 - k\frac{1}{8} = 0 \quad \Rightarrow \quad k = 32$$

(As a side remark, the units of k are lb/ft). There is no damping,

$$\frac{1}{8}u'' + 32u = 2\cos(3t) \quad u(0) = \frac{1}{6} \quad u'(0) = 0$$

(We could solve it using Method of Undetermined Coefficients, but that wasn't asked in this problem).

9. No damping, and the spring constant is given as 1 lb per in, which would be interpreted as 12 lb per ft. The mass is $6/32$, and

$$\frac{6}{32}u'' + 12u = 4\cos(7t) \qquad \text{zero I.C.s}$$

Now to solve this, we might rewrite the DE as

$$u'' + 64u = \frac{64}{3}\cos(7t)$$

The homogeneous part of the solution is $C_1\cos(8t) + C_2\sin(8t)$, and using the method of undetermined coefficients, we guess the form $y_p = A\cos(7t) + B\sin(7t)$, and solve for A, B:

$$\begin{array}{rl} 64y_p & = 64A\cos(7t) + 64B\sin(7t) \\ y_p'' & = -49A\cos(7t) - 49B\sin(7t) \\ \hline \frac{64}{3}\cos(7t) & = 15A\cos(7t) + 15B\sin(7t) \end{array}$$

so $A = 64/45$ and $B = 0$. Now, the solution is

$$u(t) = C_1\cos(8t) + C_2\sin(8t) + \frac{64}{45}\cos(7t)$$

Solve using the zero ICs:

$$\begin{array}{rl} C_1 + 64/45 & = 0 \\ 8C_2 & = 0 \end{array} \quad \Rightarrow \quad u(t) = \frac{64}{45}\left(\cos(7t) - \cos(8t)\right)$$

For the graph, we can use the cosine sum formula (as in Exercises 1, 2) to get:

$$u(t) = \frac{128}{45}\sin\left(\frac{1}{2}t\right)\sin\left(\frac{15}{2}t\right)$$

This will look like "beating", where the "slow" wave has a period of $2\pi/(1/2)$ or 4π. You should be able to plot the slow wave, but overall can be a little tricky- For that, you can use a calculator or computer.

12. The equation of motion is

$$2u'' + u' + 3u = 3\cos(3t) - 2\sin(3t)$$

The "steady state" part of the solution from the text refers to the particular part of the solution. Since there is damping, we know that the homogeneous part of the solution will NOT be purely periodic, so we can use the Method of Undetermined Coefficients without multiplying by t: $u_p = A\cos(3t) + B\sin(3t)$ and substitute into the DE to solve for A, B:

$$\begin{array}{rll} 3u_p & = 3A\cos(3t)+ & 3B\sin(3t) \\ u_p' & = 3B\cos(3t)- & 3A\sin(3t) \\ 2u_p'' & = -18A\cos(3t)- & 18B\sin(3t) \\ \hline 3\cos(3t) - 2\sin(3t) & = (-15A + 3B)\cos(3t)+ & (-3A - 15B)\sin(3t) \end{array}$$

so that (Cramer's rule is OK here):

$$\begin{array}{rl} -15A + 3B & = 3 \\ -3A - 15B & = -2 \end{array} \quad \Rightarrow \quad A = -\frac{39}{234} = -\frac{1}{6} \qquad B = \frac{1}{6}$$

The particular part of the solution is then

$$u_p(t) = \frac{1}{6}\left(\sin(3t) - \cos(3t)\right)$$

The amplitude is $\sqrt{2}/6$ and the phase shift is $\delta = tan^{-1}(-1)$.

NOTE: The angle δ should be in quadrant II (cosine is negative, sine is positive), but the inverse tangent will return an angle in quadrant IV, so add π. That is,

$$\delta = \tan^{-1}(-1) = -\frac{\pi}{4} + \pi = \frac{3\pi}{4}$$

Therefore,

$$u_p(t) = \frac{\sqrt{2}}{6}\cos\left(3t - \frac{3\pi}{4}\right)$$

For 21-23, see the Maple file below. Think about what is happening and what we would predict about the steady state (or particular) part of the solution. Since we have

$$u'' + \frac{1}{8}u' + 4u = F(t)$$

Then, if F is periodic, then we expect that the amplitude of the steady state is going to max out when the (natural) frequency of the forcing function is approximately 2.