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where $s = 0, 1$ or 2 so that no part of y_{p_i} is part of the homogeneous solution.

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CORRECT GUESS: $y_p(t) = t(At + B)\mathrm{e}^{-t}$
Note that $t^2\mathrm{e}^{-t}$ and $t\mathrm{e}^{-t}$ are not fcns in y_h.

If we continue with $y_p = (At^2 + Bt)e^{-t}$, then substitute into the DE, and divide by e^{-t}, we get the following equation, *true for all t*:

$$-(10At + (5B - 2A)) = 3t$$

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So we equate the coefficients and solve:

$$\begin{array}{lrl} t \text{ terms:} & -10A & = 3 \\ \text{Constants:} & 2A - 5B & = 0 \end{array} \quad \Rightarrow \quad \begin{array}{rl} A & = -3/10 \\ B & = -3/25 \end{array}$$

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Put it all together at the end. The general solution is

$$y(t) = C_1 e^{4t} + e^{-t}\left(C_2 - \frac{3}{25}t - \frac{3}{10}t^2\right)$$

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Give the form of y_p in each case:

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- $y'' - 4y' + 4y = e^{-2t}$ with $r = -2, -2$.

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- $y'' - 4y' + 4y = \mathrm{e}^{-2t}$ with $r = -2, -2$.

$$y_p = t^2 A\mathrm{e}^{-2t}$$

3.6: Variation of Parameters

We're looking for a technique for getting the particular solution to the *GENERAL* linear second degree ODE:

$$y'' + p(t)y' + q(t)y = g(t) \qquad (*)$$

IDEA: Let y_1, y_2 be a fundamental set of solutions to $(*)$.
ANSATZ:

$$y_p = u_1(t)y_1(t) + u_2(t)y_2(t) = u_1y_1 + u_2y_2$$

so that:

$$y_p' = (u_1'y_1 + u_1y_1') + (u_2'y_2 + u_2y_2')$$

We will assume the following, which becomes one of our conditions:

$$u_1'y_1 + u_2'y_2 = 0$$

With that, we have:

$$\begin{aligned} y_p &= u_1 y_1 + u_2 y_2 \\ y_p' &= u_1 y_1' + u_2 y_2' \\ y_p'' &= u_1' y_1' + u_1 y_1'' + u_2' y_2 + u_2 y_2'' \end{aligned}$$

Substitute into the DE: Substitute into the DE:

$$\begin{array}{rcllll} y_p'' & = & u_1'y_1'+ & u_1y_1''+ & u_2'y_2+ & u_2y_2'' \\ +p(t)y_p' & = & & pu_1y_1'+ & & pu_2y_2' \\ +q(t)y_p & = & & qu_1y_1+ & & qu_2y_2 \\ \hline g(t) & = & & & & \end{array}$$

With that, we have:

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$$\begin{array}{rlllll} y_p'' & = u_1' y_1' + & u_1 y_1'' + & u_2' y_2 + & u_2 y_2'' \\ +p(t) y_p' & = & p u_1 y_1' + & & p u_2 y_2' \\ +q(t) y_p & = & q u_1 y_1 + & & q u_2 y_2 \\ \hline g(t) & = u_1' y_1' + & 0 + & u_2' y_2' + & 0 \end{array}$$

Summary Page: Variation of Parameters

Given $y'' + p(t)y' + q(t)y = g(t)$, let y_1, y_2 solve the homogeneous equation, and set

$$y_p = u_1 y_1 + u_2 y_2$$

Then u_1, u_2 satisfy the following equations

$$\begin{aligned} u_1' y_1 + u_2' y_2 &= 0 \\ u_1' y_1' + u_2' y_2' &= g(t) \end{aligned}$$

Which is solved via Cramer's Rule:

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ g(t) & y_2' \end{vmatrix}}{W(y_1, y_2)} \quad u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & g(t) \end{vmatrix}}{W(y_1, y_2)}$$

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Which gives us u_1, u_2

$$u_1 = \int \frac{-y_2 g(t)}{W(y_1, y_2)}\,dt \quad u_2 = \int \frac{y_1 g(t)}{W(y_1, y_2)}\,dt$$

So that, if we wanted to write $y_p(t)$, we could:

$$y_p(t) = y_1(t) \int \frac{-y_2 g(t)}{W(y_1, y_2)}\,dt + y_2(t) \int \frac{y_1 g(t)}{W(y_1, y_2)}\,dt$$

Use any method to find the general solution:

$$y'' - 2y' + y = \frac{\mathrm{e}^t}{1+t^2} + t^2\mathrm{e}^t$$

SOLUTION: First, the solution to the characteristic equation is $r = 1, 1$:

$$y_h = \mathrm{e}^t(C_1 + C_2 t)$$

And...?

For $g_1(t) = \mathrm{e}^t/(1+t^2)$, use Variation of Parameters:

$$y_1 = \mathrm{e}^t, \quad y_2 = t\mathrm{e}^t, \quad W = \mathrm{e}^{2t}$$

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Therefore, the full solution is:

$$y_{p_1}(t) = u_1 y_1 + u_2 y_2 = -\frac{1}{2}\mathrm{e}^t \ln(1+t^2) + t\mathrm{e}^t \tan^{-1}(t)$$

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BUT we have to multiply by t^2, so that

$$y_p = t^2(At^2 + Bt + C)\mathrm{e}^t = (At^4 + Bt^3 + Ct^2)\mathrm{e}^t$$

A little messy algebra:

$$\begin{array}{rlllll}
+y_p & = \mathrm{e}^t(At^4 & +Bt^3 & +Ct^2) & & \\
-2(y_p' & = \mathrm{e}^t(At^4 & +(4A+B)t^3 & +(3B+C)t^2 & +2Ct) & \\
y_p'' & = \mathrm{e}^t(At^4 & +(8A+B)t^3 & +(12A+6B+C)t^2 & +(6B+4C)t & +2C \\
\hline
t^2\mathrm{e}^t & = \mathrm{e}^t(0 & +0 & +12At^2 & +6Bt & +2C)
\end{array}$$

Therefore, $A = 1/12, B = 0, C = 0$, and $y_p = \frac{1}{12}t^4\mathrm{e}^t$.

The general solution to $y'' - 2y' + y = \frac{e^t}{1+t^2} + t^2 e^t$ is:

$$y(t) = e^t \left(C_1 + C_2 t + \frac{1}{12} t^4 \right) - \frac{1}{2} e^t \ln(1 + t^2) + t e^t \tan^{-1}(t)$$