Selected Solutions, 5.4

Section 5.4 offers us a review of Euler Equations and introduces us to something called a regular singular point.
HOMEWORK CHANGE: We probably won't cover 5.5 for the exam (to be announced in class), so in that case, change the homework in Section 5.4 to the following:

$$1, 3, 5, 8, 10, 17, 22, 25, 26$$

1. $$x^2y'' + 4xy' + 2y = 0$$

   The characteristic equation is:

   $$r(r-1) + 4r + 2 = 0 \quad \Rightarrow \quad r^2 + 3r + 2 = 0 \quad \Rightarrow \quad (r+2)(r+1) = 0$$

   So the general solution is:

   $$y = C_1x^{-1} + C_2x^{-2}$$

3. $$x^2y'' - 3xy' + 4y = 0$$

   As before, go to the characteristic equation:

   $$r(r-1) - 3r + 4 = 0 \quad \Rightarrow \quad r^2 - 4r + 4 = 0 \quad \Rightarrow \quad (r-2)^2 = 0$$

   The solution is:

   $$y = x^2(C_1 + C_2 \ln|x|)$$

10. This one is a bit different. You can let $w = x - 2$, then back-substitute at the end.

    $$w^2y'' + 5wy' + 8y = 0 \quad \Rightarrow \quad r(r-1) + 5r + 8 = 0 \quad \Rightarrow \quad r^2 + 4r + 8 = 0$$

    Complete the square (or use the quadratic formula):

    $$(r^2 + 4r + 4) + 4 = 0 \quad \Rightarrow \quad (r+2)^2 = -4 \quad \Rightarrow \quad r = -2 \pm 2i$$

    Therefore, the solution is:

    $$y = x^{-2}\left(C_1 \cos(2\ln|x-2|) + C_2 \sin(2\ln|x-2|)\right)$$

17. Put the DE in standard form first:

    $$y'' + \frac{1-x}{x}y' + \frac{x}{x}y = 0$$

    Therefore, $q(x) = \frac{1-x}{x}$ and $r(x) = 1$. The point of concern is $x_0 = 0$, so we take that limit:

    $$\lim_{x\to 0} \frac{1-x}{x} = -1 \qquad\qquad \lim_{x\to 0} x^2 \cdot 1 = 0$$

    both limits are finite, so $x_0 = 0$ is a regular singular point.

22. Same technique as 17.

$$y'' + \frac{x}{x^2} + \frac{x^2 - \nu^2}{x^2} y = 0$$

so

$$\lim_{x \to 0} xp(x) = 1 \qquad \lim_{x \to 0} x^2 r(x) = -\nu^2$$

Therefore, $x_0 = 0$ is a regular singular point.

25. We have two singular points. One at $x_0 = -2$ and $x_0 = 1$. We check both.

$$y'' + \frac{3(x-1)}{(x+2)^2(x-1)} y' - \frac{2(x+2)}{(x+2)^2(x-1)}$$

At $x_0 = -2$, we have:

$$\lim_{x \to -2} (x+2)p(x) = \frac{3}{(x+2)} = \text{ DNE}$$

Since the limit does not exist, the point $x_0 = -2$ is an irregular singular point. Checking at $x_0 = 1$, we have:

$$\lim_{x \to 1} (x-1)p(x) = \frac{3(x-1)}{(x+2)^2} = 0 \qquad \lim_{x \to 1} (x-1)^2 q(x) = \frac{-2(x-1)}{(x+2)} = 0$$

Therefore, $x = 1$ is a regular singular point.

(26 is very similar to the other two problems).