Help on section 6.2

The main ideas:

- Take the Laplace transform of a DE.
- Solve it for $Y(s)$ (the transform of $y(t)$).
- Invert the transform- This may involve partial fractions and/or completing the square.

The first set of exercises focuses on this last part, where partial fractions and/or completing the square will be performed in order to use the table to do the inverse transform.

2. Table entry 11.

4. Use Partial Fractions:

$$\frac{3s}{s^2-s-6} = \frac{A}{s+2} + \frac{B}{s-3}$$

so that

$$A(s-3) + B(s+2) = 3s$$

If $s = 3$, then $B = 9/5$. If $s = -2$, then $A = 6/5$. Now,

$$\mathcal{L}^{-1}\left(\frac{6}{5}\frac{1}{s+2} + \frac{9}{5}\frac{1}{s-3}\right) = \frac{6}{5}\mathcal{L}^{-1}\left(\frac{1}{s+2}\right) + \frac{9}{5}\mathcal{L}^{-1}\left(\frac{1}{s-3}\right) = \frac{6}{5}\mathrm{e}^{-2t} + \frac{9}{5}\mathrm{e}^{3t}$$

You may use Wolfram Alpha to check your answer:

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inverse laplace transform of 3s/(s^2-s-6)
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8. Use Partial fractions:

$$\frac{8s^2-4s+12}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4}$$

Therefore,

$$8s^2-4s+12 = A(s^2+4) + (Bs+C)s \quad \Rightarrow \quad 8s^2-4s+12 = (A+B)s^2 + Cs + 4A$$

Therefore,

$$\begin{aligned} A+B &= 4 \\ C &= -4 \\ 4A &= 12 \end{aligned} \quad \Rightarrow \quad A = 3, C = -4, B = 1$$

And we get:

$$\frac{8s^2-4s+12}{s(s^2+4)} = \frac{3}{s} + \frac{5s-4}{s^2+4}$$

For the inverse Laplace transform, we look at this as:

$$3\,\frac{1}{s}+5\,\frac{s}{s^2+4}-2\,\frac{2}{s^2+4}$$

so that our final answer is:

$$3+5\cos(2t)-2\sin(2t)$$

In Wolfram Alpha, you can ask for just the partial fraction decomposition:

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partial fraction (8s^2-4s+12)/(s(s^2+4))
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10. Complete the square in the denominator:

$$\frac{2s-3}{s^2+2s+1+9}=\frac{2(s+1)-5}{(s+1)^2+9}=2\frac{s+1}{(s+1)^2+3^2}-\frac{5}{3}\frac{3}{(s+1)^2+3^2}$$

12. Take the Laplace transform of both sides:

$$s^2Y-sy(0)-y'(0)+3(sY-y(0))+2Y=0$$

Apply initial conditions and solve for $Y(s)$:

$$Y(s)=\frac{s+3}{s^2+3s+2}=\frac{2}{s+1}+\frac{1}{s+2}$$

so that

$$y(t)=2\mathrm{e}^{-t}-\mathrm{e}^{-2t}$$

14. Same idea:

$$s^2Y-sy(0)-y'(0)-4(sY-y(0))+4Y=0$$

Apply initial conditions and solve for $Y(s)$:

$$Y(s)=\frac{s-3}{(s^2-4s+4}=\frac{(s-3)}{(s-2)^2}=\frac{(s-2)-1}{(s-2)^2}=\frac{1}{s-2}-\frac{1}{(s-2)^2}$$

Use table entries 2 and 11:

$$y(t)=\mathrm{e}^{2t}-t\mathrm{e}^{2t}$$

20. Continuing in the same fashion as before, you should find that

$$Y(s)=\frac{s}{(s^2+\omega^2)(s^2+4)}+\frac{s}{s^2+\omega^2}$$

Use Partial Fractions on the first term:

$$\frac{s}{(s^2+\omega^2)(s^2+4)}=\frac{As+B}{s^2+\omega^2}+\frac{Cs+D}{s^2+4}$$

so that (multiply it out and equate coefficients):

$$\begin{array}{c|rl} s^3 & A+C & =0 \\ s^2 & B+D & =0 \\ s & 4A+C\omega^2 & =1 \\ \text{const} & 4B+D\omega^2 & =0 \end{array} \quad \Rightarrow \quad A=\frac{1}{4-\omega^2}, C=-A, B=0, D=0$$

22. Same idea as before.

$$Y(s)=\frac{1}{(s+1)(s^2-2s+2)}+\frac{1}{s^2-2s+2}$$

Using partial fractions again, get that:

$$Y(s)=\frac{1/5}{s+1}-\frac{1}{5}\cdot\frac{s-3}{s^2-2s+2}+\frac{1}{s^2-2s+2}$$

Now complete the square in the denominator of the last two terms.

24. For the Laplace transform of the right hand side of the given differential equation, define $f(t)$ as the piecewise defined function. Then:

$$\int_0^\infty e^{-st} f(t)\,dt=\int_0^\pi e^{-st}\,dt+\int_\pi^\infty 0\,dt=-\left.\frac{e^{-st}}{s}\right|_0^\pi=\frac{1-e^{-\pi s}}{s}$$

Therefore, the Laplace transform will be:

$$Y(s)=\frac{1-e^{-\pi s}}{s(s^2+4)}+\frac{s}{s^2+4}$$

28.

$$F'(s)=\frac{d}{ds}\left(\int_0^\infty e^{-st}f(t)\,dt\right)=\int_0^\infty \frac{d}{ds}\left(e^{-st}\right)f(t)\,dt=\int_0^\infty -te^{-st}f(t)\,dt=\mathcal{L}(-tf(t))$$

29. Use Exercise 28, which in this case is

$$-\mathcal{L}(-te^{at})=-F'(s)$$

where $F(s)=\frac{1}{s-a}$. Therefore,

$$-\mathcal{L}(-te^{at})=\frac{1}{(s-a)^2}$$

As in table entry 11 with $n=1$.

30. Same idea as Exercise 29, except we differentiate twice (a little messy) to get:

$$\mathcal{L}(t^2\sin(bt)) = \frac{d^2}{ds^2}\left(\frac{b}{s^2+b^2}\right) = \frac{d}{ds}\left((-2bs)(s^2+b^2)^{-2}\right) = \frac{6bs^2-2b^3}{(s^2+b^2)^3}$$

33. Differentiate table entry 9 with respect to s, then multiply by -1.

37. This one is a bit tricky (A Challenge Problem!)

$$\mathcal{L}(g(t)) = \int_0^\infty \mathrm{e}^{-st}\left[\int_0^t f(w)\,dw\right]\,dt = \int_0^\infty \int_0^t \mathrm{e}^{-st} f(w)\,dw\,dt$$

In the (t,w) plane, we have $0 \le w \le t$ and $0 \le t \le \infty$. If we reverse the order of integration then $t \ge w$ and $0 \le w \le \infty$ (draw it in the (t,w) plane). Therefore,

$$\int_0^\infty f(w)\left[\int_w^\infty \mathrm{e}^{-st}\right]\,dt\,dw = \int_0^\infty f(w)\frac{\mathrm{e}^{-sw}}{s}\,dw$$

which is what we wanted: $\frac{1}{s}\mathcal{L}(f(t))$.