Quiz 10 solutions

1. If $\lambda = 2+3i$ and $\mathbf{v} = \begin{bmatrix} 1+i \\ 1 \end{bmatrix}$, find the real part and the imaginary part of the vector: $\mathrm{e}^{\lambda t}\mathbf{v}$.

   SOLUTION:

$$\mathrm{e}^{(2+3i)t} \begin{bmatrix} 1+i \\ 1 \end{bmatrix} = \mathrm{e}^{2t}(\cos(3t) + i\sin(3t)) \begin{bmatrix} 1+i \\ 1 \end{bmatrix} =$$

$$\mathrm{e}^{2t} \begin{bmatrix} (\cos(3t) - \sin(3t)) + i(\cos(3t) + \sin(3t)) \\ \cos(3t) + i\sin(3t) \end{bmatrix}$$

   The real and imaginary parts are the vectors below (respectively):

$$\mathrm{Real}(\mathrm{e}^{\lambda t}\mathbf{v}) = \mathrm{e}^{2t} \begin{bmatrix} (\cos(3t) - \sin(3t)) \\ \cos(3t) \end{bmatrix} \qquad \mathrm{Imag}(\mathrm{e}^{\lambda t}\mathbf{v}) = \mathrm{e}^{2t} \begin{bmatrix} (\cos(3t) + \sin(3t)) \\ \sin(3t) \end{bmatrix}$$

2. Give the general solution (you may leave it implicit) to the following ODE by first writing dy/dx:

$$\frac{dx}{dt} = x - y$$
$$\frac{dy}{dt} = y - 4x$$

   SOLUTION:

$$\frac{dy}{dx} = \frac{y-4x}{x-y} = \frac{\frac{y}{x} - 4}{1 - \frac{y}{x}}$$

   Let $v = y/x$, so that $y' = v'x + v$, and

$$v'x + v = \frac{v-4}{1-v} \quad \Rightarrow \quad v'x = \frac{v-4}{1-v} - v = \frac{v - 4 - v(1-v)}{1-4} = \frac{v^2-4}{1-v}$$

   Therefore,

$$\int \frac{1-v}{v^2-4}\, dv = \int \frac{1}{x}\, dx$$

   On the left side, use partial fractions so that we get

$$-\frac{3}{4}\int \frac{dv}{v+2} - \frac{1}{4}\int \frac{dv}{v-2} = \int \frac{1}{x}\, dx$$

   Multiplying by -4 and integrating, we get ($x > 0$):

$$3\ln(v+2) + \ln(v-2) = -4\ln(x) + C \quad \Rightarrow \quad 3\ln(y/x+2) + \ln(y/x-2) = -4\ln(x) + C$$

3. Find the eigenvalues and eigenvectors for the matrix A:

(a) $A = \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix}$

SOLUTION: The characteristic equation is

$$\lambda^2 - \lambda - 6 = 0 \quad \Rightarrow \quad (\lambda - 3)(\lambda + 2) = 0$$

For $\lambda_1 = -2$, we find an eigenvector:

$$\begin{aligned} (2+2)v_1 - 4v_2 &= 0 \\ -v_1 + (-1+2)v_2 &= 0 \end{aligned} \quad \Rightarrow \quad v_1 = v_2 \quad \Rightarrow \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Similarly, for $\lambda_2 = 3$, we find an eigenvector:

$$\begin{aligned} (2-3)v_1 - 4v_2 &= 0 \\ -v_1 + (-1-3)v_2 &= 0 \end{aligned} \quad \Rightarrow \quad v_1 = -4v_2 \quad \Rightarrow \quad \mathbf{v}_2 = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$$

(b) $A = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix}$

SOLUTION: The characteristic equation is $\lambda^2 + 1 = 0$, so $\lambda = \pm i$.
If $\lambda = i$, we solve for an eigenvector:

$$\begin{aligned} (2-i)v_1 - 5v_2 &= 0 \\ v_1 + (-2-i)v_2 &= 0 \end{aligned} \quad \Rightarrow \quad v_1 = (2+i)v_2 \quad \Rightarrow \quad \mathbf{v} = \begin{bmatrix} 2+i \\ 1 \end{bmatrix}$$

Similarly, for $\lambda = -i$, we have:

$$\begin{aligned} (2+i)v_1 - 5v_2 &= 0 \\ v_1 + (-2+i)v_2 &= 0 \end{aligned} \quad \Rightarrow \quad v_1 = (2-i)v_2 \quad \Rightarrow \quad \mathbf{v} = \begin{bmatrix} 2-i \\ 1 \end{bmatrix}$$

We see that an eigenvector for the complex conjugate of λ is the complex conjugate of the vector.

4. Solve the following system using eigenvalues and eigenvectors:

$$\begin{aligned} x_1' &= -2x_1 + x_2 \\ x_2' &= x_1 - 2x_2 \end{aligned}$$

SOLUTION: You should find the eigenvalues and eigenvectors are, respectively:

$$\lambda_{1,2} = -3, -1 \qquad \mathbf{v}_{1,2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The solution is therefore

$$\mathbf{x}(t) = C_1 \mathrm{e}^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + C_2 \mathrm{e}^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$