There is one point in the trace-determinant plane where many different possibilities arise. If the trace and determinant are both zero, the chart shows that our system can change into any type of system whatsoever. All three of the critical loci meet at this point.

It is helpful to think of these three critical loci as fences. As long as we change parameters so that (T, D) does not pass over one of the fences, the linear system remains "unchanged" in the sense that the qualitative behavior of the solutions does not change. However, passing over a fence changes the behavior dramatically. The system undergoes a bifurcation.

A One-Parameter Family of Linear Systems

Consider the one-parameter family of linear systems $d\mathbf{Y}/dt = \mathbf{AY}$, where

$$\mathbf{A} = \begin{pmatrix} -2 & a \\ -2 & 0 \end{pmatrix},$$

which depends on the parameter a. As a varies, the determinant of this matrix is $2a$, but the trace is always -2. If we vary the parameter a from a large negative number to a large positive number, the corresponding point (T, D) in the trace-determinant plane moves vertically along the straight line $T = -2$ (see Figure 3.50). As a increases, we first travel from the saddle region into the region where we have a real sink. This change occurs when the system admits a zero eigenvalue, which in turn occurs at $a = 0$. As a continues to increase, we next move across the repeated-root parabola, and the system changes from having a sink with real eigenvalues to a spiral sink. This second bifurcation occurs when $T^2 - 4D = 0$, which for this example reduces to $D = 1$. Hence this bifurcation occurs at $a = 1/2$.

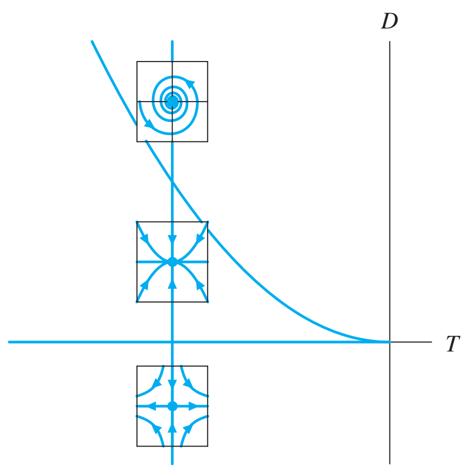

Figure 3.50
Motion in the trace-determinant plane corresponding to the one-parameter family of systems

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}, \quad \text{where } \mathbf{A} = \begin{pmatrix} -2 & a \\ -2 & 0 \end{pmatrix}.$$

Bifurcation from sink to spiral sink

Let's investigate how the bifurcation from sink to spiral sink occurs in terms of the phase portraits of the corresponding systems. We need first to compute the eigenvalues and eigenvectors of the system. Of course these quantities depend on a. Since the characteristic polynomial is $\lambda^2 + 2\lambda + 2a = 0$, the eigenvalues are

$$\lambda = \frac{-2 \pm \sqrt{4 - 8a}}{2} = -1 \pm \sqrt{1 - 2a}.$$

As we deduced above, if $a > 1/2$, then $1 - 2a < 0$ and the eigenvalues are complex with negative real part. For $a < 1/2$, the eigenvalues

$$\lambda = -1 \pm \sqrt{1 - 2a}$$

are both real. In particular, if $0 < a < 1/2$, $\sqrt{1 - 2a} < 1$, so both eigenvalues are negative. Hence the origin is a sink with two straight lines of solutions (see Figure 3.51).

If we compute the eigenvectors for the eigenvalue $\lambda = -1 + \sqrt{1 - 2a}$, we find that they lie along the line

$$y = \left(\frac{1 + \sqrt{1 - 2a}}{a} \right) x.$$

Similarly, the eigenvectors corresponding to the eigenvalue $\lambda = -1 - \sqrt{1 - 2a}$ lie along the line

$$y = \left(\frac{1 - \sqrt{1 - 2a}}{a} \right) x.$$

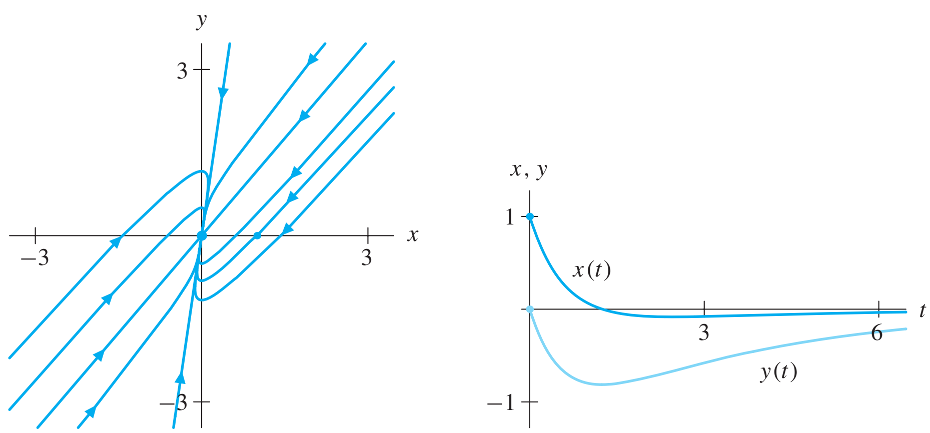

Figure 3.51
Phase portrait and the $x(t)$- and $y(t)$-graphs for the indicated solution for the one-parameter family with $a = 1/4$.