Summary of Beating and Resonance

The Equations

We consider oscillators with no damping and periodic forcing, and we think of the forcing as having an adjustable period (we can change ω, but b is fixed).

$$y'' + b^2 y = \cos(\omega t) \qquad y(0) = 0, \quad y'(0) = 0$$

Initially, we assume $\omega \neq b$. The full solution to the IVP was then:

$$y(t) = \frac{1}{b^2 - \omega^2}(\cos(\omega t) - \cos(bt))$$

It is possible to write this expression as:

$$y(t) = \frac{1}{b^2 - \omega^2}\left(-2\sin\left(\frac{\omega + b}{2}t\right)\sin\left(\frac{\omega - b}{2}\,t\right)\right)$$

And when $\omega = b$, the solution becomes: $y(t) = \frac{1}{2b} t \sin(bt)$

The Analysis

When considering solutions to the undamped mass-spring model with periodic forcing, we found that, as we vary the frequency of the forcing function, as it gets close to the natural (homogeneous) solution, then **beating** will begin to occur. As the forcing frequency matches the natural frequency, the forced response will "blow up", which is **resonance**.

We also found that the beat period (for the "slow" function) is half the period of the slower sine:

$$\frac{1}{2} \cdot \frac{2\pi}{\frac{\omega - b}{2}} = \frac{\pi}{\frac{\omega - b}{2}} = \frac{2\pi}{\omega - b}$$

and the beat amplitude was:

$$A = \frac{2}{|b^2 - \omega^2|}$$

The period of the "fast" beat is the period of $\sin((\omega + b)t/2)$.

These things together say that, as $\omega \to b$, the amplitude and the period of the beats get larger, and larger and larger- Finally blowing up when they are equal.

For example, if $\omega = 2.17$ and $b = 2$, then we'll get beating, with:

$$\text{Period of one beat } \approx \frac{2\pi}{0.17} \approx 11.76\pi$$

and the amplitude of the beat is: $\frac{2}{|b^2 - \omega^2|} \approx \frac{2}{0.71} \approx 2.82$.

Something to consider: In nature, there is never the situation where there is no damping at all, but there may be damping that is so small as to be basically 0.