Clarification of a couple of Points in 4.4

In Section 4.4, our text makes several statements that we wish to clarify here, and a caution.

CAUTION: In Section 4.4, our text is comparing the steady state to the form: $A\cos(\omega t+\phi)$ and NOT to the form we used in class (and will continue to use):

$$R\cos(\omega t - \delta)$$

This does not effect the amplitude, but it does effect the phase shift.

On page 420, replace the formula for ϕ with the formula for δ:

$$\tan(\phi) = -\frac{p\omega}{q-\omega^2} \quad \Rightarrow \quad \tan(\delta) = \frac{p\omega}{q-\omega^2}$$

where we add π if needed (so δ will be from Quadrant I or II).

Additional Note:

With the ansatz $y_p = A\mathrm{e}^{i\omega t}$, we solved $y'' + py' + qy = cos(\omega t)$ by finding that

$$A = \frac{1}{b} = \frac{1}{(q-\omega^2) + pwi}$$

When looking for the maximum amplitude of A with respect to ω, remember that

$$\frac{d}{d\omega}\frac{1}{\sqrt{f(\omega)}} = \frac{f'(\omega)}{-2(f(\omega)^{3/2}}$$

So if we set that to zero, we just need to find $f'(\omega) = 0$.

Example 1

Find the particular solution to $y'' + y' + 4y = \cos(2t)$.

Solution:

The ansatz is $y_p = A\mathrm{e}^{2it}$, and substituting it into the DE gives us:

$$A\mathrm{e}^{2it}\left(-4 + 2i + 4\right) = \mathrm{e}^{i\omega t}$$

so that $A = \frac{1}{2i}$. The amplitude is $1/|2i|$, or $1/2$.

The phase angle is the argument of $2i$, which is $\pi/2$.

$$\frac{1}{2}\cos\left(2t - \frac{\pi}{2}\right) = \frac{1}{2}\cos\left(2\left(t - \frac{\pi}{4}\right)\right)$$

Example 2

Compute the amplitude and phase shift for the steady state solution to

$$y'' + 4y' + 4y = 2\cos(3t)$$

SOLUTION:

Complexify so that we have the following, and the steady state solution is the real part of the solution to:

$$y'' + 4y' + 4y = 2e^{3ti}$$

Substitute Ae^{3it} into the DE to get:

$$Ae^{3it}(-9 + 12i + 4) = 2e^{3it} \quad \Rightarrow \quad A = \frac{2}{-5 + 12i}$$

The amplitude of the steady state is the amplitude of A:

$$A = \frac{2}{\sqrt{25 + 144}} = \frac{2}{\sqrt{169}} = \frac{2}{13}$$

The phase shift of the steady state is the polar angle of b (if $A = 1/b$), where we add π since the point $(-5, 12)$ is in Quadrant II:

$$\delta = \tan^{-1}\left(-\frac{12}{5}\right) + \pi \approx 1.966 \text{ rad}$$

Example 3

Suppose that we have the damped oscillator model given by:

$$y'' + 2y' + 5y = \cos(\omega t)$$

Find the value of ω that maximize the amplitude of the response.

SOLUTION:

Ansatz is $y_p = Ae^{i\omega t}$, so that $y_p' = i\omega Ae^{i\omega t}$ and $y_p'' = -\omega^2 Ae^{i\omega t}$:

$$Ae^{i\omega t}(-\omega^2 + 2\omega i + 5) = e^{i\omega t} \quad \Rightarrow \quad A = \frac{1}{(5 - \omega^2) + 2\omega i} \quad \Rightarrow \quad |A| = \frac{1}{\sqrt{(5 - \omega^2)^2 + 4\omega^2}}$$

When we differentiate, we only need to work with what is inside the radical:

$$2(5 - \omega^2)(-2\omega) + 8\omega = 0 \quad \Rightarrow \quad \omega = \sqrt{3}$$

We've shown that the amplitude of the response is maximized when the circular frequency of the forcing function is given by $\omega = \sqrt{3}$.

Practice Problems, Section 4.4

Section 4.4 doesn't give you very many numerical problems, so try out the ones below to help you with the ideas and notation.

1. Find the amplitude and phase angle for the particular solution to: $y''+2y'+3y = \cos(2t)$
2. Find the amplitude and phase angle for the particular solution to: $y''+3y'+9y = \sin(2t)$
3. Find the particular solution to $y'' + 2y' + y = \cos(3t)$.
4. Find the particular solution to $y'' + \frac{1}{10}y' + 4y = \cos(2t)$
5. Show that, if $A = \frac{1}{\alpha+\beta i}$, then the real part of $Ae^{i\omega t}$ is given by:

$$\frac{\alpha}{\alpha^2+\beta^2}\cos(\omega t) + \frac{\beta}{\alpha^2+\beta^2}\sin(\omega t)$$

6. Use the shortcut from the previous exercise to find the particular solution to

$$y'' + 3y' + 4y = \cos(3t)$$

7. Recall that given $y'' + py' + qy = \cos(\omega t)$, and the ansatz $y_p = Ae^{i\omega t}$, we found that

$$A = \frac{1}{\sqrt{(q-\omega^2)^2 + p^2\omega^2}}.$$

   (a) Verify that the formula for A is correct (by substituting y_p into the DE).
   (b) Find a simplified expression for $\frac{dA}{d\omega} = 0$. Remember to use our hint from class.

8. Given $y'' + y' + 4y = \cos(\omega t)$, find the value of ω that will maximize the amplitude of the response.