Practice Problems, Section 4.4

Section 4.4 doesn't give you very many numerical problems, so try out the ones below to help you with the ideas and notation.

1. Find the amplitude and phase angle for the particular solution to: $y''+2y'+3y = \cos(2t)$

   SOLUTION: Start with the ansatz $y_p = A\mathrm{e}^{2it}$.

   If we only need the amplitude and phase angle, the problem can be solved pretty quickly once we have A. Substituting y_p into the DE, and factoring $A\mathrm{e}^{2it}$ out of the left side, we have:

   $$A\mathrm{e}^{2it}(-4+2(2i)+3) = \mathrm{e}^{2it} \quad \Rightarrow \quad A = \frac{1}{-1+4i}$$

   Therefore, the amplitude of the particular part of the solution is:

   $$A = \frac{1}{\sqrt{(-1)^2+4^2}} = \frac{1}{\sqrt{17}}$$

   and the phase angle:

   $$\delta = \tan^{-1}\left(\frac{4}{-1}\right) + \pi$$

   If you're doing this by hand, you can leave your answer as that.

   Extra Note: If you were to multiply it out and simply, you would find that

   $$y_p(t) = -\frac{1}{17}\cos(2t) + \frac{4}{17}\sin(2t)$$

2. Find the amplitude and phase angle for the particular solution to: $y''+3y'+9y = \sin(2t)$

   SOLUTION: Same idea as before, but now we would look at the imaginary part to e^{2it} for the full solution. The amplitude doesn't change, but the angle does change. To see this, we'll go ahead and do it long-hand:

   $$A\mathrm{e}^{2it}(-4+3(2i)+9) = \mathrm{e}^{2it} \quad \Rightarrow \quad A = \frac{1}{5+6i}$$

   If we rationalize the denominator of A, we get

   $$A = \frac{5}{25+36} - \frac{6}{25+36}\,i = \frac{5}{61} - \frac{6}{61}\,i$$

   The full particular solution is the imaginary part of $A\mathrm{e}^{2it}$:

   $$\mathrm{Im}\left[\left(\frac{5}{61} - \frac{6}{61}\,i\right)(\cos(2t)+i\sin(2t))\right] = -\frac{6}{61}\cos(2t) + \frac{5}{61}\sin(2t)$$

We see that the amplitude is what we expect, $1/\sqrt{61}$. For the angle, notice that $\frac{-6}{61} + \frac{5}{61}\, i$ is in Quadrant II, so we'll need to add π:

$$\delta = \tan^{-1}\left(-\frac{5}{6}\right) + \pi$$

Now we have written y_p as:

$$y_p(t) = \frac{1}{\sqrt{61}} \cos(2t - \delta)$$

(with δ as computed in the line before).

3. Find the particular solution to $y'' + 2y' + y = \cos(3t)$.

   SOLUTION: We're back to standard form, but now we want the full particular solution. As usual, take the ansatz and substitute into the DE, getting:

   $$A\mathrm{e}^{3it}(-9 + 2(3i) + 1) = \mathrm{e}^{3it} \quad \Rightarrow \quad A = \frac{1}{-8 + 6i}$$

   The amplitude is $1/\sqrt{100} = 1/10$ and $\delta = \tan^{-1}(-6/8) + \pi \approx 2.498$, so

   $$y_p = \frac{1}{10} \cos(3t - 2.498)$$

   It would be OK to give your solution as a sum as well, since it wasn't specified:

   $$y_p = \frac{-8}{100}\cos(3t) + \frac{6}{100}\sin(3t) = \frac{2}{25}\cos(3t) + \frac{3}{50}\sin(3t)$$

4. Find the particular solution to $y'' + \frac{1}{10}y' + 4y = \cos(2t)$

   SOLUTION: We've done a few of these now. The equation we get is:

   $$A\mathrm{e}^{2it}\left(-4 + \frac{1}{10}(2i) + 4\right) = \mathrm{e}^{2it} \quad \Rightarrow \quad A = \frac{5}{i} = -5i$$

   We could also write $A = 1/(i/5)$ to make it look like $1/b$. In this case, the amplitude is $1/|b| = 5$ and $\delta = \pi/2$.

   $$y_p = 5\cos(2t - \pi/2) = 5\sin(2t)$$

   NOTE: You could write the solution either way.

5. Show that, if $A = \frac{1}{\alpha + \beta i}$, then the real part of $A\mathrm{e}^{i\omega t}$ is given by:

   $$\frac{\alpha}{\alpha^2 + \beta^2}\cos(\omega t) + \frac{\beta}{\alpha^2 + \beta^2}\sin(\omega t)$$

   SOLUTION: This is straightforward by multiplying everything out.

6. Use the shortcut from the previous exercise to find the particular solution to

$$y'' + 3y' + 4y = \cos(3t)$$

SOLUTION: Substitute the ansatz and get the value of A:

$$A\,(-9 + 3(3i) + 4) = 1 \quad \Rightarrow \quad A = \frac{1}{-5 + 9i}$$

From our previous computation, the solution is:

$$y_p = \frac{-5}{106}\cos(3t) + \frac{9}{106}\sin(3t)$$

7. Recall that given $y'' + py' + qy = \cos(\omega t)$, and the ansatz $y_p = Ae^{i\omega t}$, we found that

$$A = \frac{1}{\sqrt{(q - \omega^2)^2 + p^2\omega^2}}.$$

(a) Verify that the formula for A is correct (by substituting y_p into the DE).
SOLUTION: CAUTION: This isn't the constant A we're used to seeing- It is the **amplitude** of y_p, so this really ought to say that:

$$R = \frac{1}{\sqrt{(q - \omega^2)^2 + p^2\omega^2}}$$

With that corrected, we can find R easily by computing the usual version of A:

$$Ae^{i\omega t}(-\omega^2 + p\omega i + q) = e^{i\omega t} \Rightarrow A = \frac{1}{(q - \omega^2) + p\omega i} \Rightarrow R = \frac{1}{|b|} = \frac{1}{\sqrt{(q - \omega^2)^2 + p^2\omega^2}}$$

(b) Find a simplified expression for $\frac{dA}{d\omega} = 0$. Remember to use our hint from class.
SOLUTION: In class, we said that, if

$$F(x) = \frac{1}{\sqrt{f(x)}}$$

then the critical points of F are found by:

$$F'(x) = 0 \quad \Rightarrow \quad \frac{-1}{2}(f(x))^{-3/2}f'(x) = 0 \quad \Rightarrow \quad \frac{f'(x)}{-2(f(x))^{3/2}} = 0$$

And multiplying both sides by the denominator, we only need to compute

$$f'(x) = 0$$

In this case, $f(\omega) = (q - \omega^2)^2 + p^2\omega^2$, and

$$f'(\omega) = 2(q - \omega^2)(-2\omega) + 2p^2\omega = 0$$

Our critical value of ω is then:

$$\omega = \sqrt{q - \frac{p^2}{2}}$$

8. Given $y'' + y' + 4y = \cos(\omega t)$, find the value of ω that will maximize the amplitude of the response.

SOLUTION: Using our previous computation, we have $p = 1, q = 4$, so

$$\omega = \sqrt{4 - \frac{1}{2}} \approx 1.87$$

For a graph of the amplitude, R, versus ω, see below (not required, but nice to see).