NOTES from Section 3.6

Case with one real eigenvalue

In the case where λ is real and a repeated root, how would we write the solution in terms of the initial condtions (so that the form of the solution matches what we did before).

In that case,

$$y(t) = C_1 e^{\lambda t} + C_2 t e^{\lambda t} \quad \Rightarrow \quad y_0 = c_1$$

So far, we have

$$y(t) = y_0 e^{\lambda t} + C_2 t e^{\lambda t}$$

Solving for C_2, we need $y'(0) = v_0$, so:

$$y'(t) = y_0 \lambda e^{\lambda t} + C_2 e^{\lambda t} + \lambda C_2 t e^{\lambda t}$$

so that

$$v_0 = y_0 \lambda + C_2 \quad \Rightarrow \quad C_2 = -y_0 \lambda + v_0$$

This way, the solution can be expressed as:

$$y(t) = e^{\lambda t} \left(y_0 + t(v_0 - \lambda y_0) \right)$$

Comparing to the System

Putting the equation in the form of a system,

$$ay'' + by' + cy = 0 \quad \Rightarrow \quad \mathbf{Y}' = \begin{bmatrix} 0 & 1 \\ -c/a & -b/a \end{bmatrix} \mathbf{Y}$$

From the system perspective, our solution would be:

$$\mathbf{Y}(t) = e^{\lambda t} \mathbf{V}_0 + t e^{\lambda t} \mathbf{V}_1$$

where the first part of the vector (the one we want) is:

$$\mathbf{V}_1 = \begin{bmatrix} -\lambda & 1 \\ \dots & \dots \end{bmatrix} \begin{bmatrix} y_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} -\lambda y_0 + v_0 \\ \dots \end{bmatrix}$$

And we see that we get the same constant as before.

Summary of the Comparison

When we're working with single second order DEs, it is probably more straightforward to write the general solution (in the case of a single eigenvalue) as:

$$y(t) = e^{\lambda t}(C_1 + C_2 t)$$

For the system of equations, it is easier to write the general solution as:

$$\mathbf{Y}(t) = e^{\lambda t} \left[\mathbf{V}_0 + t \mathbf{V}_1 \right]$$

where $\mathbf{V}_0 = \mathbf{Y}(0)$ and $\mathbf{V}_1 = (A - \lambda I)\mathbf{V}_0$.