

## Review Questions, Exam 2

1. A weight of 4 lbs is attached to the end of a spring that is then stretched 3 inches. The spring is then attached to a dashpot that provides 6 lbs of resistance for each ft/s of velocity. If the spring is pushed up 6 inches and released, give the second order linear differential equation that models the position of the mass.

SOLUTION: Recall that  $4 = mg$ , so for the spring constant, we take  $mg - kL = 0$  (Remember to convert  $L$  to feet!):

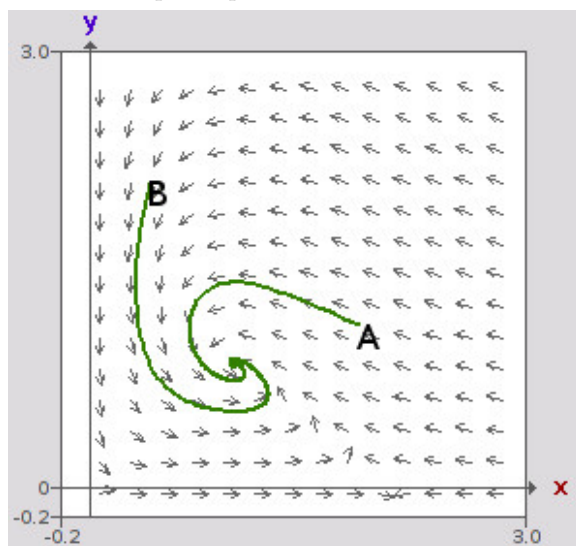
$$4 - \frac{k}{4} = 0 \Rightarrow k = 16$$

We're given the damping coefficient of 6, and  $mg = 4$ , so  $m = 4/g = 4/32 = 1/8$ :

$$\frac{1}{8}y'' + 6y' + 16 = 0 \quad y(0) = -\frac{1}{2}, \quad y'(0) = 0$$

2. Consider the system below, with two solution curves shown in the phase plane.

$$\begin{aligned} \frac{dx}{dt} &= x \left( 2 - x - \frac{6}{5}y \right) \\ \frac{dy}{dt} &= y(-1 + x) \end{aligned}$$



- (a) Find the equilibrium solutions.

SOLUTION: From the first equation, we have

$$x = 0 \quad \text{or} \quad x = 2 - \frac{6}{5}y$$

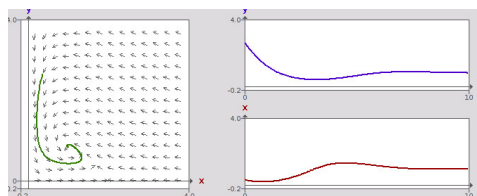
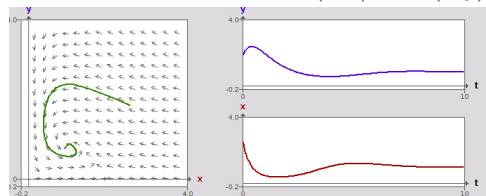
If  $x = 0$  from Equation 1, then Equation 2 becomes  $y = 0$ , so  $(0, 0)$  is one equilibrium solution.

If  $x = 2 - \frac{6}{5}y$  from Equation 1, then Equation 2 becomes:

$$y \left( -1 + \left( 2 - \frac{6}{5}y \right) \right) \Rightarrow y = 0 \text{ or } y = \frac{5}{6}$$

This gives  $(2, 0)$  and  $(1, 5/6)$  as the second and last equilibria.

- (b) Given the phase plane from `HPGSystemSolver` with starting points  $A$  and  $B$  as shown above, sketch the solution in the  $(t, x)$  and  $(t, y)$  planes.



3. Consider the system of differential equations:  $\mathbf{Y}' = \begin{bmatrix} 0 & 4 \\ -1 & 0 \end{bmatrix} \mathbf{Y}$

(a) Find the eigenvalues to this system.

SOLUTION: From  $\lambda^2 + 4 = 0$ , we get  $\lambda = \pm 2i$

(b) Find the general solution of this system.

SOLUTION: We need an eigenvector first, and we'll use  $\lambda = 2i$ :

$$-2iv_1 + 4v_2 = 0 \Rightarrow \mathbf{v} = \begin{bmatrix} 4 \\ 2i \end{bmatrix} \text{ or } \begin{bmatrix} 2 \\ i \end{bmatrix} \Rightarrow e^{\lambda t} \mathbf{v} = (\cos(2t) + i \sin(2t)) \begin{bmatrix} 2 \\ i \end{bmatrix}$$

Multiplying this out, we get:

$$e^{\lambda t} \mathbf{v} = \begin{bmatrix} 2 \cos(2t) + 2i \sin(2t) \\ -\sin(2t) + i \cos(2t) \end{bmatrix}$$

The general solution is then:

$$\mathbf{Y}(t) = C_1 \begin{bmatrix} 2 \cos(2t) \\ -\sin(2t) \end{bmatrix} + C_2 \begin{bmatrix} 2 \sin(2t) \\ \cos(2t) \end{bmatrix}$$

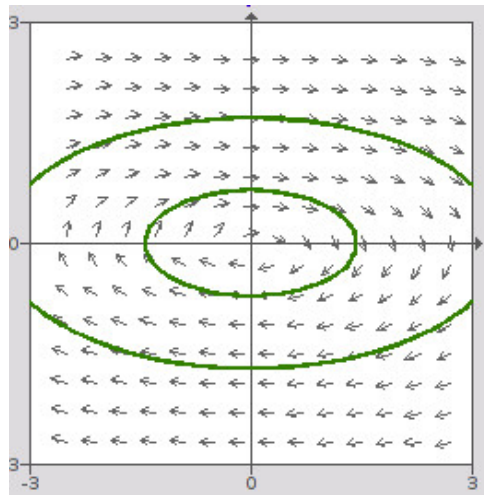
(c) Find the solution that satisfies  $\mathbf{Y}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

SOLUTION: In this case, the equations are easy enough to solve directly. On the exam, you might need to use Cramer's Rule (it's good to remember it!)

$$\begin{array}{rcl} 2C_1 & +0C_2 & = 0 \\ 0C_1 & +C_2 & = 1 \end{array} \Rightarrow \mathbf{Y}(t) = \begin{bmatrix} 2 \sin(2t) \\ \cos(2t) \end{bmatrix}$$

(d) Sketch the phase plane for this system.

SOLUTION: The sketch is of a **center**.



4. Consider the family of systems of differential equations given by:

$$(a) \quad \mathbf{Y}' = \begin{bmatrix} a & 1 \\ -1 & a \end{bmatrix} \mathbf{Y} \quad (b) \quad \mathbf{Y}' = \begin{bmatrix} a & 1 \\ a & a \end{bmatrix} \mathbf{Y}$$

Use the Poincare classification to determine how the equilibrium changes. Begin by drawing the trace-determinant plane, then track where you are on the graph.

- (a) For (a),  $\text{Tr}(A) = 2a$ ,  $\det(A) = a^2 + 1$  and  $\Delta = (2a)^2 - 4(a^2 + 1) = -1$ .

Therefore, the determinant is always positive and the discriminant is always negative- We are inside the parabola, and the classification depends only on the trace:

- If  $a > 0$ , the origin is a SPIRAL SOURCE.
- If  $a = 0$ , the origin is a CENTER.
- If  $a < 0$ , the origin is a SPIRAL SINK.

- (b) For (b),  $\text{Tr}(A) = 2a$ ,  $\det(A) = a^2 - a = a(a - 1)$  and  $\Delta = (2a)^2 - 4(a^2 - a) = a$

Now, the position depends on the signs of each of these quantities. One way to track them is to use a number line.

|            |         |             |         |
|------------|---------|-------------|---------|
| $2a$       | $-$     | $+$         | $+$     |
| $a(a - 1)$ | $+$     | $-$         | $+$     |
| $a$        | $-$     | $+$         | $+$     |
|            | $a < 0$ | $0 < a < 1$ | $a > 1$ |

We see that if  $a < 0$ , then we are in the upper left quadrant, inside the parabola (SPIRAL SINK). we note that if  $a = 0$ , then everything is zero, and we have "UNIFORM MOTION". If  $0 < a < 1$ , the determinant is negative, and the origin is a SADDLE. If  $a = 1$ , we have a line of stable fixed points, and finally if  $a > 1$ , we have a SOURCE.

5. Given the mass-spring system with unit mass ( $m = 1$ ), spring constant  $k > 0$  and damping  $b \geq 0$ , write the second order linear homogeneous differential equation describing the motion of the mass.

SOLUTION:  $y'' + by' + ky = 0$

- (a) Convert the differential equation into a system of first order equations in the form  $\mathbf{Y}' = A\mathbf{Y}$  (that is, find  $A$ ).

$$\mathbf{Y}' = \begin{bmatrix} 0 & 1 \\ -k & -b \end{bmatrix} \mathbf{Y}$$

- (b) Continuing with the previous question, find the eigenvalues of  $A$ .

SOLUTION: The characteristic equation is  $\lambda^2 + b\lambda + k = 0$ , so the eigenvalues are given by the quadratic formula:

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4k}}{2}$$

- (c) Continuing, state all the values of  $b, k$  so that the system has complex eigenvalues.

SOLUTION: To have complex eigenvalues, the discriminant must be negative:  $b^2 - 4k < 0$ , or  $b^2 < 4k$ .

- (d) State all the values of  $b, k$  for which the system has a saddle at the origin.

SOLUTION: To have a saddle at the origin, the determinant must be negative, or  $k < 0$ . However, this is a spring constant, so  $k$  will never be negative. Therefore, the origin is never a saddle.

6. Below is a coupled pair of second order equations in  $x, y$ . Convert it to a linear system of four first order equations. You might let  $u_1, u_2, u_3, u_4$  be your variables.

$$\begin{aligned} x'' + 2x' - y' + x + 4y &= 0 \\ y'' + y' + 8y - 3x &= 0 \end{aligned}$$

SOLUTION: Let  $u_1 = x$ ,  $u_2 = x'$ ,  $u_3 = y$  and  $u_4 = y'$ . We then write the system of DEs for  $u_i$ :

$$\begin{aligned} u_1' &= u_2 \\ u_2' &= x'' = -x - 4y - 2x' + y' = -u_1 - 2u_2 - 4u_3 + u_4 \\ u_3' &= u_4 \\ u_4' &= y'' = 3x - 8y - y' = 3u_1 - 8u_3 - u_4 \end{aligned}$$

It's not necessary, but we could write this system as:

$$\mathbf{u}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -2 & -4 & 1 \\ 0 & 0 & 0 & 1 \\ 3 & 0 & -8 & -1 \end{bmatrix} \mathbf{u}$$

7. Solve:  $\mathbf{Y}' = A\mathbf{Y}$ ,  $\mathbf{Y}(0)$  shown.

(a)  $A = \begin{bmatrix} -4 & 1 \\ 2 & -3 \end{bmatrix}$ ,  $\mathbf{Y}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ . Sketch the phase plane for this one.

SOLUTION:  $Tr(A) = -7$  and  $\det(A) = 10$ , so  $\lambda^2 + 7\lambda + 10 = 0$  gives us  $(\lambda + 2)(\lambda + 5) = 0$ , or  $\lambda = -2, -5$  (the origin is a SINK). Now compute eigenvectors:

- For  $\lambda = -2$ , we have  $(-4 - (-2))v_1 + v_2 = 0$ , or  $-2v_1 + v_2 = 0$ .  
One eigenvector would be  $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ .
- For  $\lambda = -5$ , we have  $(-4 - (-5))v_1 + v_2 = 0$ , or  $v_1 + v_2 = 0$ .  
One eigenvector would be  $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ .

The general solution is now

$$\mathbf{Y}(t) = C_1 e^{-2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 e^{-5t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

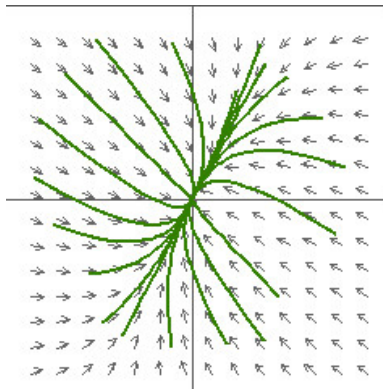
To get the initial condition,

$$\begin{array}{rcl} C_1 - C_2 & = & 1 \\ 2C_1 + C_2 & = & 1 \end{array} \Rightarrow C_1 = \frac{\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix}} = \frac{2}{3}, \quad C_2 = \frac{\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix}} = \frac{-1}{3}$$

The general solution is now

$$\mathbf{Y}(t) = \frac{2}{3} e^{-2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \frac{1}{3} e^{-5t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

For the phase plane, I would be looking for your eigenvectors and the solution curves should bend towards the first eigenvector (the one with the “not-so-negative” exponent).



(b)  $A = \begin{bmatrix} -4 & 4 \\ -1 & 0 \end{bmatrix}$ ,  $\mathbf{Y}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ .

We follow the same procedure. In this case,  $\text{Tr}(A) = -4$  and  $\det(A) = 4$ , so that

$$\lambda^2 + 4\lambda + 4 = 0 \quad \Rightarrow \quad (\lambda + 2)^2 = 0 \quad \Rightarrow \quad \lambda = -2, -2$$

We have a double eigenvalue. If we solved for the eigenvectors, we would just get one:  $(-4 + 2)v_1 + 4v_2 = 0$ , or  $-v_1 + 2v_2 = 0$ . But its not needed this time. We just need to solve for  $\mathbf{v}_1$ , which we said would be:

$$\begin{array}{rcl} (-4 + 2)(1) + 4(2) & = & v_{11} \\ -1(1) + (0 + 2)(2) & = & v_{12} \end{array} \quad \Rightarrow \quad \mathbf{v}_1 = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$$

The solution is:

$$\mathbf{Y}(t) = e^{-2t} \left( \begin{bmatrix} 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 6 \\ 3 \end{bmatrix} \right)$$