Study Guide: Exam 1, Math 244

The exam covers material from Chapters 1 and will be 50 minutes in length, which covers material about the general first order differential equation. You may not use the text, notes, colleagues or a calculator.

Calculus Review

Please be sure you're familiar with basic "integration by parts" and "partial fractions".

Vocabulary

- You should know what these terms mean:

  differential equation, ordinary differential equation, partial differential equation, order of a differential equation, initial value problem, particular solution (vs general solution), equilibrium solution, direction field (or slope field).

- Understand what it means for a given function to be a *solution* to a DE, and be able to verify that a given function is a solution to a given DE.
- Be able to identify the following types of DEs: Linear, separable, homogeneous, autonomous (these are not mutually exclusive).

The Existence and Uniqueness Theorem

Given the IVP: $y' = f(t, y)$, (t_0, y_0):

Let the functions f and f_y be continuous in some open rectangle R containing the point (t_0, y_0). Then there exists an interval about t_0, $(t_0 - h, t_0 + h)$ contained in R for which a unique solution to the IVP exists.

Side Remark 1: To determine such a time interval, we must solve the DE.

Side Remark 2: We broke out the theorem in class into two components (existence and uniqueness). You can use either the theorem there or as it stated above.

Graphical Analysis

1. Be able to use a direction field to analyze the behavior of solutions to general first order equations. Be able to construct simple direction fields using isoclines (an isocline is a curve where the value of the derivative is constant).
2. Special Case: **Autonomous DEs:** The main idea here is to be able to graph the phase plot, $y' = f(y)$ in the (y, y') plane and be able to translate the information from this graph to the direction field, the (t, y) plane (and vice-versa). We also constructed "phase lines".

   Here is a summary of that information:

In Phase Diagram:	In Direction Field:
y intercepts	Equilibrium Solutions
$+$ to $-$ crossing	equil. is SINK
$-$ to $+$ crossing	equil. is SOURCE
$+$ to $+$ or $-$ to $-$	equil is NODE
$y' > 0$	y increasing
$y' < 0$	y decreasing
y' and df/dy same sign	y is concave up
y' and df/dy mixed	y is concave down

Recall that we also looked at a theorem about determining the stability of an equilibrium solution using the sign of df/dy, and determining a formula for concavity given $y' = f(y)$- That was:

$$\frac{d^2y}{dt^2} = \frac{df}{dy}\frac{dy}{dt} = \frac{df}{dy}\,f(y)$$

Analytic Solutions

- Linear: $y' = a(t)y + b(t)$ or $y' + g(t)y = b(t)$.

  The general solution is found by solving for both the homogeneous part of the solution (when $b(t) = 0$), $y_h(t)$, and a particular part of the solution, $y_p(t)$. Then the general solution is given by:

  $$y(t) = Cy_h(t) + y_p(t)$$

  We had two methods- Guess and check, and the integrating factor:

  - Guess and check:
    - The autonomous part is $y_h(t) = \mathrm{e}^{\int a(t)\,dt}$.
    - For the particular part, if $b(t)$ is an exponential, polynomial or sine/cosine (or a product of these), then guess the same general form. If your guess matches the homogeneous solution, multiply the guess by t.
  - Use the integrating factor: $\mu(t) = \mathrm{e}^{\int g(t)\,dt}$

    $$\mu(t)(y' + g(t)y) = \mu b(t) \quad \Rightarrow \quad (\mu y)' = \mu b \quad \Rightarrow \quad muy = \int \mu b(t)\,dt$$

    then solve for $y(t)$.

- Separable: $y' = f(y)g(t)$. Separate variables: $(1/f(y))\,dy = g(t)\,dt$

Numerical Methods

Be able to describe and write the formula for Euler's Method. Be able to compute a solution using Euler's Method (for simple time steps).

Solve by Substitution

This refers to Appendix A. There would be a suggested substitution for you.

Models

Recall that the words: "A is proportional to B" is generally interpreted mathematically as: $A = kB$.

A list of the models we've thought about:

- Population Models:
  - Exponential growth (or decay)
  - Logistic growth
- Newton's Law of Cooling
- Tank Mixing
- Compound interest

We won't consider electrical circuits (at least for the time being).

Review Questions

Our textbook has an excellent set of review questions on pg 136-141. In particular, look at the following. NOTE: The modeling questions may have a numerical part that require a calculator to solve - I won't be asking you questions that require a calculator.

- # 1-20
- # 21-39
- 40(a)
- 41, 42, 45, 48
- 49, 50, 52
- 53, 54

In Appendix A, I would give you a suggested substitution, like exercises 1-4, 8-10, in 23-26, these are Bernoulli equations.