Exam 2 Summary

Notes

The exam will cover material from Section 3.1 to 3.8. Recall that, for Variation of Parameters, the system of equations will be provided to you:

$$\begin{aligned} u_1' y_1 + u_2' y_2 &= 0 \\ u_1' y_1' + u_2' y_2' &= g(t) \end{aligned}$$

So you'll need to remember what all the notation meant (in terms of the original problem).

Structure and Theory (Mostly 3.2)

The goal of the theory was to establish the structure of solutions to the second order IVP:

$$y'' + p(t)y' + q(t)y = g(t), \quad y(t_0) = y_0$$

We saw that two functions form a fundamental set of solutions to the homogeneous DE if the Wronskian is not zero at t_0.

1. Vocabulary: Linear operator, general solution, fundamental set of solutions, linear combination.

2. Theorems:

   - Abel's Theorem.
     If y_1, y_2 are solutions to $y'' + p(t)y' + q(t)y = 0$, then the Wronskian, $W(y_1, y_2)$, is either always zero or never zero on the interval for which the solutions are valid.
     That is because the Wronskian may be computed as:
     $$W(y_1, y_2)(t) = C\mathrm{e}^{-\int p(t)\,dt}$$
   - The Structure of Solutions to $y'' + p(t)y' + q(t)y = g(t)$, $y(t_0) = y_0, y'(t_0) = v_0$
     Given a fundamental set of solutions to the homogeneous equation, y_1, y_2, then there is a solution to the initial value problem, written as:
     $$y(t) = C_1 y_1(t) + C_2 y_2(t) + y_p(t)$$
     where $y_p(t)$ solves the non-homogeneous equation.
     In fact, if we have: $y'' + p(t)y' + q(t)y = g_1(t) + g_2(t) + \ldots + g_n(t)$,, we can solve by splitting the problem up into smaller problems:
     - y_1, y_2 form a fundamental set of solutions to the homogeneous equation.
     - y_{p_1} solves $y'' + p(t)y' + q(t)y = g_1(t)$
     - y_{p_2} solves $y'' + p(t)y' + q(t)y = g_2(t)$
       and so on..
     - y_{p_n} solves $y'' + p(t)y' + q(t)y = g_n(t)$

     and the full solution is: $y(t) = C_1 y_1 + C_2 y_2 + y_{p_1} + y_{p_2} + \ldots + y_{p_n}$.

Finding the Homogeneous Solution

We had two distinct equations to solve-

$$ay'' + by' + cy = 0 \qquad \text{or} \qquad y'' + p(t)y' + q(t)y = 0$$

First we look at the case with constant coefficients, then we look at the more general case.

Constant Coefficients

To solve

$$ay'' + by' + cy = 0$$

we use the **ansatz** $y = \mathrm{e}^{rt}$. Then we form the associated **characteristic equation**:

$$ar^2 + br + c = 0 \qquad \Rightarrow \qquad r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

so that the solutions depend on the discriminant, $b^2 - 4ac$ in the following way:

- $b^2 - 4ac > 0 \Rightarrow$ two distinct real roots r_1, r_2. The general solution is:

  $$y_h(t) = c_1\mathrm{e}^{r_1 t} + c_2\mathrm{e}^{r_2 t}$$

  If $a, b, c > 0$ (as in the Spring-Mass model) we can further say that r_1, r_2 are negative. We would say that this system is OVERDAMPED.

- $b^2 - 4ac = 0 \Rightarrow$ one real root $r = -b/2a$. Then the general solution is:

  $$y_h(t) = \mathrm{e}^{-(b/2a)t}\,(C_1 + C_2 t)$$

  If $a, b, c > 0$ (as in the Spring-Mass model), the exponential term has a negative exponent. In this case (one real root), the system is CRITICALLY DAMPED.

- $b^2 - 4ac < 0 \Rightarrow$ two complex conjugate solutions, $r = \alpha \pm i\beta$. Then the solution is:

  $$y_h(t) = \mathrm{e}^{\alpha t}\,(C_1 \cos(\beta t) + C_2 \sin(\beta t))$$

  If $a, b, c > 0$, then $\alpha = -(b/2a) < 0$. In the case of complex roots, the system is said to the UNDERDAMPED. If $\alpha = 0$ (this occurs when there is no damping), we get pure periodic motion, with period $2\pi/\beta$ or circular frequency β.

Solving the more general case

We had two methods for solving the more general equation:

$$y'' + p(t)y' + q(t)y = 0$$

but each method relied on already having one solution, $y_1(t)$. Given that situation, we can solve for y_2 (so that y_1, y_2 form a fundamental set), by one of two methods:

- By use of the Wronskian: There are two ways to compute this,
  - $W(y_1, y_2) = C\mathrm{e}^{-\int p(t)\,dt}$ (This is from Abel's Theorem)
  - $W(y_1, y_2) = y_1 y_2' - y_2 y_1'$

  Therefore, these are equal, and y_2 is the unknown: $y_1 y_2' - y_2 y_1' = C\mathrm{e}^{-\int p(t)\,dt}$
- Reduction of order, where $y_2 = v(t)y_1(t)$. Now substitute y_2 into the DE, and use the fact that y_1 solves the homogeneous equation, and the DE reduces to:

$$y_1 v'' + (2y_1' + py_1)v' = 0$$

Finding the particular solution.

Our two methods were: Method of Undetermined Coefficients and Variation of Parameters.

Method of Undetermined Coefficients

This method is motivated by the observation that, a linear operator of the form $L(y) = ay'' + by' + cy$, acting on certain classes of functions, returns the same class. In summary, the table from the text:

if $g_i(t)$ is:	The ansatz y_{p_i} is:
$P_n(t)$	$t^s(a_0 + a_1 t + \ldots a_n t^n)$
$P_n(t)\mathrm{e}^{\alpha t}$	$t^s\mathrm{e}^{\alpha t}(a_0 + a_1 t + \ldots + a_n t^n)$
$P_n(t)\mathrm{e}^{\alpha t}\sin(\mu t)$ or $\cos(\mu t)$	$t^s\mathrm{e}^{\alpha t}\left((a_0 + a_1 t + \ldots + a_n t^n)\sin(\mu t) + (b_0 + b_1 t + \ldots + b_n t^n)\cos(\mu t)\right)$

The t^s term comes from an analysis of the homogeneous part of the solution. That is, multiply by t or t^2 so that no term of the ansatz is included as a term of the homogeneous solution.

Variation of Parameters:

Given $y'' + p(t)y' + q(t)y = g(t)$, with y_1, y_2 solutions to the homogeneous equation, we write the ansatz for the particular solution as: $y_p = u_1 y_1 + u_2 y_2$. Substitute this in, and from our analysis, we saw that u_1, u_2 were required to solve the following, which can be done using Cramer's Rule:

$$\begin{aligned} u_1' y_1 + u_2' y_2 &= 0 \\ u_1' y_1' + u_2' y_2' &= g(t) \end{aligned} \quad \Rightarrow \quad u_1' = \frac{-y_2 g}{W(y_1, y_2)} \qquad u_2' = \frac{y_1 g}{W(y_1, y_2)}$$

Analysis of the Oscillator Model (3.7-3.8)

Given

$$mu'' + \gamma u' + ku = F(t)$$

we should be able to determine the constants from a given setup for a spring-mass system. Once that's done, be able to analyze the spring-mass system in some particular cases:

1. Unforced (The homogeneous equation, $F(t) = 0$)
   (a) No damping: Natural frequency is $\sqrt{k/m}$
   (b) With damping: Underdamped, Critically Damped, Overdamped
2. Periodic Forcing[1]
   (a) With no damping: Determine when Beating and Resonance occur.

   $$u'' + \omega^2 u = F\cos(\omega_0 t)$$

   "Beating" occurs when ω is close to ω_0.
   The circular frequency for one beat is $|\omega_0 - \omega|$. The amplitude of one beat: $2F/(\omega_0^2 - \omega^2)$.
   "Resonance" occurs when $\omega = \omega_0$. Resonance forces the solution to become unbounded (can be very bad in the physical world!)

   (b) With damping: We changed the model to make our computations a bit easier:

   $$u'' + pu' + qu = \cos(\omega t)$$

   Then with $y_p = Ae^{i\omega t}$, we found that

   $$A = \frac{1}{(q - \omega^2) + ip\omega} = \frac{1}{\alpha + \beta i}$$

   Given this, we found that the amplitude R and phase angle δ of the forced response (also known as the particular part of the solution) is given by:

   $$R = \frac{1}{|\alpha + \beta i|} \qquad \delta = \tan^{-1}(\beta/\alpha)$$

   Given this, be able to determine q or ω that will maximize the amplitude R (by differentiating, setting to zero.) Using that value of ω in the physical world can result in Resonance (blowing up the wine glass or a bridge!)

[1] The more general case of forcing we would use the Method of Undetermined Coefficients to solve.