Summary- Elements of Chapters 7

We started with some basic matrix algebra- Be sure you know how to perform matrix-vector multiplication and matrix-matrix multiplication for 2×2 matrices.

Eigenvalues and Eigenvectors

1. Definition: Given an $n \times n$ matrix A, if there is a constant λ and a non-zero vector $\mathbf{v}$ so that

$$A\mathbf{v} = \lambda\mathbf{v} \quad \text{or} \quad \begin{array}{ll} av_1 + bv_2 & = \lambda v_1 \\ cv_1 + dv_2 & = \lambda v_2 \end{array}$$

   then λ is an eigenvalue, and $\mathbf{v}$ is an associated eigenvector.

2. To find λ, first we form the main set of equations (subtract λ from the previous equations):

$$\begin{array}{lll} av_1 & +bv_2 & = \lambda v_1 \\ cv_1 & +dv_2 & = \lambda v_2 \end{array} \Leftrightarrow \begin{array}{rrl} (a-\lambda)v_1 & +bv_2 & = 0 \\ cv_1 & +(d-\lambda)v_2 & = 0 \end{array} \tag{1}$$

   This system has a non-zero solution for v_1, v_2 only if the determinant of coefficients is 0:

$$\det \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} = 0 \quad \Rightarrow \quad \lambda^2 \Leftrightarrow \lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = 0$$

   where $\mathrm{Tr}(A)$ is the trace of A (which we defined as $a + d$). For each λ, we must go back and solve Equation (1). This equation is the characteristic equation.

3. Remember that using our λ, the system of equations should be two lines that are multiples of each other. That means the solution is the line represented by (using the first equation) by the following (as long as both values are not zero)

$$(a-\lambda)v_1 + bv_2 = 0 \quad \rightarrow \quad \mathbf{v} = \begin{bmatrix} -b \\ a-\lambda \end{bmatrix}$$

Solve $\mathbf{Y}' = A\mathbf{Y}$

1. Our initial ansatz was the following, from which we got eigenvalues and eigenvectors:

$$\mathbf{Y}(t) = \mathrm{e}^{\lambda t}\mathbf{v} = \mathrm{e}^{\lambda t}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{\lambda t}v_1 \\ \mathrm{e}^{\lambda t}v_2 \end{bmatrix}$$

2. The type of solution depends on what kinds of solutions we get from the (quadratic) characteristic equation. That depends on the discriminant from the quadratic formula.

$$\lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = 0 \qquad \lambda = \frac{\mathrm{Tr}(A) \pm \sqrt{\Delta}}{2}$$

   The solution is one of three cases, depending on $\Delta = \mathrm{Tr}(A)^2 - 4\det(A)$:

- Real λ_1, λ_2 with two eigenvectors, $\mathbf{v}_1, \mathbf{v}_2$:

$$\mathbf{Y}(t) = C_1 \mathrm{e}^{\lambda_1 t}\mathbf{v}_1 + C_2 \mathrm{e}^{\lambda_2 t}\mathbf{v}_2$$

- Complex $\lambda = a + ib$, $\mathbf{v}$ (we only need one):

$$\mathbf{Y}(t) = C_1 \mathrm{Re}\left(\mathrm{e}^{\lambda t}\mathbf{v}\right) + C_2 \mathrm{Im}\left(\mathrm{e}^{\lambda t}\mathbf{v}\right)$$

  Computational Note: As in Chapter 3, our solutions here are real solutions- That means you should not have an i in your final answer.

- One eigenvalue, one eigenvector. In this case, take $\mathbf{v}$ to be your initial condition, $\mathbf{v} = (x_0, y_0)$ and then determine vector $\mathbf{w}$ as:

$$\begin{aligned} (a-\lambda)v_1 + bv_2 &= w_1 \\ cv_1 + (d-\lambda)v_2 &= w_2 \end{aligned}$$

  Notice that if the initial condition is not given, you can use x_0, y_0 in place of v_1, v_2. The solution is then

$$\mathbf{Y}(t) = \mathrm{e}^{\lambda t}\left(\mathbf{v} + t\mathbf{w}\right)$$

  (This is NOT a general solution- This is a specific solution using your initial condition)

You might find this helpful- Below there is a chart comparing the solutions from earlier second order to the solutions now:

	Second Order	Systems
Form:	$ay'' + by' + cy = 0$	$\mathbf{Y}' = A\mathbf{Y}$
Ansatz:	$y = \mathrm{e}^{rt}$	$\mathbf{Y} = \mathrm{e}^{\lambda t}\mathbf{v}$
Char Eqn:	$ar^2 + br + c = 0$	$\lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = 0$
Real Solns	$y = C_1\mathrm{e}^{r_1 t} + C_2\mathrm{e}^{r_2 t}$	$\mathbf{Y}(t) = C_1\mathrm{e}^{\lambda_1 t}\mathbf{v}_1 + C_2\mathrm{e}^{\lambda_2 t}\mathbf{v}_2$
Complex	$y = C_1\mathrm{Re}(\mathrm{e}^{rt}) + C_2\mathrm{Im}(\mathrm{e}^{rt})$	$\mathbf{Y}(t) = C_1\mathrm{Re}\left(\mathrm{e}^{\lambda t}\mathbf{v}\right) + C_2\mathrm{Im}\left(\mathrm{e}^{\lambda t}\mathbf{v}\right)$
SingleRoot	$y = \mathrm{e}^{rt}(C_1 + C_2 t)$	$\mathbf{Y}(t) = \mathrm{e}^{\lambda t}\left(\mathbf{v} + t\mathbf{w}\right)$

Classification of the Equilibria

The origin is always an equilibrium solution to $\mathbf{x}' = A\mathbf{x}$, and we can use the Poincaré Diagram to help us classify the origin. For a general nonlinear system, the equilibria are where $x' = 0$ and $y' = 0$.

Solve General Nonlinear Equations

We don't have a method that will work on every system of nonlinear differential equations, although there are some tricks we can try with special cases- that is, given the system

$$\begin{aligned} \tfrac{dx}{dt} &= f(x,y) \\ \tfrac{dy}{dt} &= g(x,y) \end{aligned} \qquad \Rightarrow \qquad \frac{dy}{dx} = \frac{g(x,y)}{f(x,y)}$$

And we might get lucky if it is in the form of an equation from Chapter 2.