Final Exam Review: Math 244

For the exam, I will provide the table of Laplace transforms (same one as before), the Poincaré diagram. The equations for Variation of Parameters will also be provided (as before).

Some of the modeling questions below may require a calculator, but I will try my best to make the numbers on the "work out nicely", so no calculator will be allowed.

You should go through the three exam study guides and question sets, the old exams, and the old quizzes. If you get stuck on anything below, you should try to go back to the book for more types of questions like that.

Students often ask about how the material will be weighted- There is some crossover between the latest material and Chapters 2 and 3, but generally speaking the exam will be weighted about 30% on the latest material, about 30% on Chapters 5 and 6, and about 40% on Chapters 2 and 3.

1. Solve (use any method if not otherwise specified):

   (a) $-t\cos(t)\,dt + (2x - 3x^2)\,dx = 0$

   (b) $y'' + 2y' + y = \sin(3t)$

   (c) $y' = y(y-1)$

   (d) $y'' - 3y' + 2y = \mathrm{e}^{2t}$

   (e) $y' = \sqrt{t}\mathrm{e}^{-t} - y$

   (f) $x' = 2 + 2t^2 + x + t^2 x$

   (g) $\begin{aligned} x_1' &= 2x_1 + 3x_2 \\ x_2' &= 4x_1 + x_2 \end{aligned}$

   (h) $(y\cos(x) + 2x\mathrm{e}^y) + (\sin(x) + x^2\mathrm{e}^y - 1)y' = 0$

2. Use the ansatz $y = t^r$ to get the general solution to the linear DE: $t^2y'' - 2ty' - 10y = 0$

3. Show that using $v = y/x$, the following equation becomes separable as a DE in v. NOTE: You do not need to solve the differential equation.

$$\frac{dy}{dx} = \frac{3x - 4y}{y - 2x}$$

4. Show that with the substitution $w = y^3$, the following equation becomes linear in w. NOTE: You do not need to solve the differential equation.

$$\frac{dy}{dx} + 3xy = \frac{x}{y^2}$$

5. Obtain the general solution in terms of α, then determine a value of α so that $y(t) \to 0$ as $t \to \infty$:

$$y'' - y' - 6y = 0, \quad y(0) = 1, y'(0) = \alpha$$

6. If $y' = y(1-y)(2-y)(3-y)(4-y)$ and $y(0) = 5/2$, determine what y does as $t \to \infty$. Hint: Do not try to actually solve the DE.

7. If y_1, y_2 are a fundamental set of solutions to $t^2y'' - 2y' + (3+t)y = 0$ and if $W(y_1, y_2)(2) = 3$, find $W(y_1, y_2)(4)$.

8. (i) What is the Wronskian? How is it used? (ii) Explain Abel's Theorem.

9. Give the two Existence and Uniqueness Theorems we have had in class (we actually had three, but list them for first order).

10. Let $y'' - 6y' + 9y = F(t)$. For each $F(t)$ listed, give the *form* of the general solution using undet. coeffs (do not solve for the coefficients).

    (a) $F(t) = 2t^2$

    (b) $F(t) = te^{-3t}\sin(2t)$

    (c) $F(t) = t\sin(2t) + \cos(2t)$

    (d) $F(t) = 2t^2 + 12e^{3t}$

11. Give Newton's law of cooling in words, then as a differential equation, then solve it!

12. A spring is stretched 0.1 m by a force of 3 N. A mass of 2 kg is hung from the spring and is also attached to a damper that exerts a force of 3 N when the velocity of the mass is 5 m/s. If the mass is pulled down 0.05 m below its resting equilibrium and released with a downward velocity of 0.1 m/s, determine its position u at time t.

13. Let $y(x)$ be a power series solution to $y'' - xy' - y = 0,\ x_0 = 1$. Find the recurrence relation and write the first 5 terms of the expansion of y.

14. Let $y(x)$ be a power series solution to $y'' - xy' - y = 0,\ x_0 = 1$ (the same as the previous DE), with $y(1) = 1$ and $y'(1) = 2$. Compute the first 5 terms of the Taylor series for the solution by computing derivatives.

15. Use the *definition* of the Laplace transform to determine $\mathcal{L}(f)$: $f(t) = \begin{cases} 3, & 0 \le t \le 2 \\ 6-t, & 2 < t \end{cases}$.

16. Determine the Laplace transform:

    (a) t^2e^{-9t}

    (b) $u_5(t)(t-2)^2$

    (c) $e^{3t}\sin(4t)$

    (d) $e^t\delta(t-3)$

17. Find the inverse Laplace transform:

    (a) $\dfrac{2s-1}{s^2-4s+6}$

    (b) $\dfrac{7}{(s+3)^3}$

    (c) $\dfrac{e^{-2s}(4s+2)}{(s-1)(s+2)}$

    (d) $\dfrac{3s-2}{(s-4)^2-3}$

18. Solve the given initial value problems using Laplace transforms:

    (a) $y'' + 2y' + 2y = 4t,\ y(0) = 0,\ \ y'(0) = -1$

    (b) $y'' - 2y' - 3y = u_1(t),\ y(0) = 0,\ \ y'(0) = -1$

    (c) $y'' - 4y' + 4y = t^2e^t,\ y(0) = 0,\ \ y'(0) = 0$ (You may write the solution as a convolution)

19. Consider $t^2y'' - 4ty' + 6y = 0$. Using $y_1 = t^2$ as one solution, find y_2 by computing the Wronskian two ways.

20. For the following differential equations, (i) Give the general solution, (ii) Solve for the specific solution, if its an IVP, (iii) State the interval for which the solution is valid.

(a) $y' - \frac{1}{2}y = e^{2t} \qquad y(0) = 1$

(b) $y' = \frac{1}{2}y(3-y)$

(c) $y'' + 2y' + y = 0, \quad y(0) = \alpha, y'(0) = 1$

(d) $2xy^2 + 2y + (2x^2y + 2x)y' = 0$

(e) $y'' + 4y = t^2 + 3e^t, y(0) = 0, y'(0) = 1.$

21. Suppose $y' = -ky(y-1)$, with $k > 0$. Sketch the phase diagram. Find and classify the equilibrium. Draw a sketch of y on the direction field, paying particular attention to where y is increasing/decreasing and concave up/down.

22. True or False (and explain): Every separable equation is also exact. If true, is one way easier to solve over the other?

23. Let $y' = 2y^2 + xy^2$, $y(0) = 1$. Solve, and find the minimum of y. Hint: Determine the interval for which the solution is valid.

24. A sky diver weighs 180 lbs and falls vertically downward. Assume that the constant for air resistance is 3/4 before the parachute is released, and 12 after it is released at 10 sec. Assume velocity is measured in feet per second, and $g = 32$ ft/sec^2.

    (a) Find the velocity of the sky diver at time t (before the parachute opens).

    (b) If the sky diver fell from an altitude of 5000 feet, find the sky diver's position at the instant the parachute is released.

    (c) After the parachute opens, is there a limiting velocity? If so, find it. (HINT: You do not need to re-solve the DE).

25. Rewrite the following differential equations as an equivalent system of first order equations. If it is an IVP, also determine initial conditions for the system.

    (a) $y'' - 3y' + 4y = 0$, $y(0) = 1$, $y'(0) = 2$.

    (b) $y''' - 2y'' - y' + 4y = 0$

    (c) $y'' - yy' + t^2 = 0$

26. Convert one of the variables in the following systems to an equivalent higher order differential equation, and solve it (be sure to solve for both x and y): $\begin{aligned} x' &= 4x + y \\ y' &= -2x + y \end{aligned}$

27. Solve the previous system by using eigenvalues and eigenvectors.

28. Verify by direct substitution that the given power series is a solution of the differential equation:

$$y = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n \qquad\qquad (x+1)y'' + y' = 0$$

29. Convert the given expression into a single power series:

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n + 2\sum_{n=2}^{\infty} n a_n x^{n-2} + 3\sum_{n=1}^{\infty} a_n x^n$$

30. Find the recurrence relation for the coefficients of the series solution to $y'' - (1+x)y = 0$ at $x_0 = 0$.

31. Find the first 5 non-zero terms of the series solution to $y'' - (1+x)y = 0$ if $y(0) = 1$ and $y'(0) = -1$ (use derivatives).

32. Let $y'' + \omega^2 y = \cos(\alpha t)$.

    (a) What values of ω, α will result in *beating*? Write the homogenous part of the solution, then give the *form* of the particular part of the solution from Method of Undetermined Coefficients.

    (b) Repeat the first part, except for *resonance*.

33. Let $y'' + \alpha y' + y = 0$. Find (all) values of α for which the solution is *underdamped*, *overdamped*, and *critically damped*.

34. Let $y'' + y' + y = \cos(2t)$.

    (a) If we complexify the problem, how is the right side of the equation changed? How is the ansatz changed?

    (b) Using your previous answer, find the amplitude and the phase shift of the forced response, $y_p(t) = R\cos(\omega t - \delta)$.

35. Given $y'' + \alpha y' + y = \cos(2t)$, find α that will maximize the amplitude of the forced response. (You might do the previous problem first!)

36. Solve, and determine how the solution depends on the initial condition, $y(0) = y_0$: $y' = 2ty^2$

37. Solve the linear system $\mathbf{x}' = A\mathbf{x}$ using eigenvalues and eigenvectors, if A is as defined below:

    (a) $A = \begin{bmatrix} 2 & 8 \\ -1 & -2 \end{bmatrix}$ (b) $A = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}$ (c) $A = \begin{bmatrix} 3 & -18 \\ 2 & -9 \end{bmatrix}$

38. For each system $\mathbf{x}' = A\mathbf{x}$, the matrix A depends on the parameter α. Find how the classification of the origin changes depending on α, indicating where the solution lies on the Poincare Diagram.

    (a) $\begin{bmatrix} 2 & -5 \\ \alpha & -2 \end{bmatrix}$ (b) $\begin{bmatrix} \alpha & 2 \\ 3 & 1 \end{bmatrix}$