**Solutions to the Homework**
**Replaces Section 3.8**

1. Solve the IVP $u'' + \omega_0^2 u = F_0\cos(\omega t)$, $u(0) = 0$ and $u'(0) = 0$, if $\omega \neq \omega_0$.

   SOLUTION: Rewriting the DE to complexify the right hand side,

   $$u'' + \omega_0^2 u = F_0(\cos(\omega)t + i\sin(\omega t)$$

   we'll solve the full problem, then (because the original function was cosine) take the real part.

   $$y_p = A\mathrm{e}^{i\omega t} \qquad y_p'' = -\omega^2\mathrm{e}^{-i\omega t} \quad \Rightarrow \quad A\mathrm{e}^{i\omega t}(-\omega^2 + \omega_0^2) = F_0\mathrm{e}^{i\omega t}$$

   Therefore, $A = F_0/(\omega_0^2 - \omega^2)$, and we want the real part of $A\mathrm{e}^{i\omega t}$:

   $$y_p = \frac{F_0}{\omega_0^2 - \omega^2}()\cos(\omega t) + i\sin(\omega t))$$

   The real part is our final answer. Including the homogenous part,

   $$u(t) = C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t) + \frac{F_0}{\omega_0^2 - \omega^2}\cos(\omega t)$$

   Putting in the initial conditions,

   $$u(t) = \frac{F_0}{\omega_0^2 - \omega^2}(\cos(\omega t) - \cos(\omega_0\, t)$$

2. Show that the period of motion of an undamped vibration of a mass hanging from a vertical spring is $2\pi\sqrt{L/g}$

   SOLUTION: With no damping, $mu'' + ku = 0$ has solution

   $$u(t) = A\cos\left(\sqrt{\frac{k}{m}}t\right) + B\sin\left(\sqrt{\frac{k}{m}}t\right)$$

   so the period is given below. We also note that $mg - kL = 0$, and this equation yields the desired substitution:

   $$P = \frac{2\pi}{\sqrt{\frac{k}{m}}} = 2\pi\sqrt{\frac{m}{k}} \qquad \text{and} \qquad mg = kL \quad \Rightarrow \quad \frac{k}{m} = \frac{g}{L}$$

3. Convert the following to $R\cos(\omega t - \delta)$

   (a) $\cos(9t) - \sin(9t)$
   In this case, $R = \sqrt{2}$ and $\delta = \tan^{-1}(-1) = -\pi/4$
   Note that in this case, we don't need to add $\pi$ because $(1, -1)$ is in Quadrant IV.

   (b) $2\cos(3t) + \sin(3t)$
   SOLUTION: $R = \sqrt{5}$ and $\omega = 3$. The angle $\delta$ is computed as the argument of the point $(2, 1)$, which you can leave as $\delta = \tan^{-1}(1/2)$:

   $$2\cos(3t) + \sin(3t) = \sqrt{5}\cos(3t - \tan^{-1}(1/2))$$

(c) $-2\pi\cos(\pi t) - \pi\sin(\pi t)$

SOLUTION: Same idea, but note that $(-2\pi, -\pi)$ is a point in Quadrant III, so we add (or subtract) $\pi$:

$$R = \pi\sqrt{5} \qquad \text{and} \qquad \delta = \tan^{-1}(1/2) + \pi \qquad \text{and} \qquad \omega = \pi$$

(d) $5\sin(t/2) - \cos(t/2)$

SOLUTION: Did you notice I reversed the sine and cosine on you (that was a mistake, but maybe it was a helpful one). The value of $R$ and *omega* would be the same either way, but $\delta$ changes:

$$R = \sqrt{26} \qquad \omega = \frac{1}{2}$$

For $\delta$, notice that our "point" is $(-1, 5)$ which is in Quadrant II, so add $\pi$:

$$\sqrt{26}\cos\left(\frac{t}{2} - \tan^{-1}(-5) - \pi\right)$$

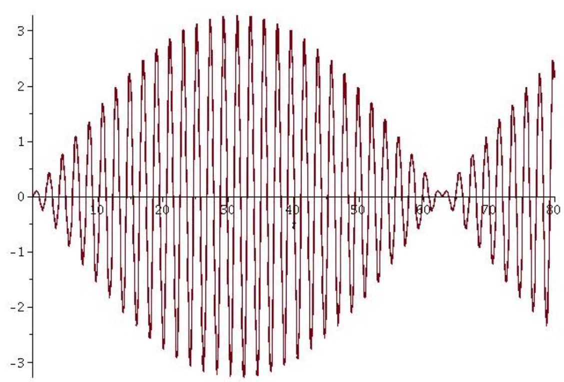
4. (Assigned as a quiz)

5. $u'' + 9u = \cos(3.1\,t)$

For this, the (circular) beat frequency is $|\omega_0 - \omega| = 1/10$ and the amplitude of a beat is $2/(\omega_0^2 - \omega^2)$, or approximately 3.28. The particular part of the solution is

$$\frac{1}{\omega_0^2 - \omega^2}\cos(\omega t) = 1.64\cos(3.1t)$$

Not necessary, but we can check the figure below provided by Maple:




6. $u'' + u = \cos(1.3\,t)$

For this, the (circular) beat frequency is $3/10$ and the amplitude of a beat is $2/(\omega_0^2 - \omega^2)$, or approximately 2.9. The particular part of the solution is

$$\frac{1}{\omega_0^2 - \omega^2}\cos(\omega t) = 1.45\cos(1.3t)$$

7. Solve $u'' + 9u = \cos(3t)$ with zero ICs.

   The solution is:
$$u(t) = \frac{1}{6} t \sin(3t)$$

8. Find the general solution of the given differential equation:
$$y'' + 3y' + 2y = \cos(t)$$

   First, get the homogeneous part of the solution by solving the characteristic equation:
$$r^2 + 3r + 2 = 0 \quad \Rightarrow \quad (r+2)(r+1) = 0 \quad \Rightarrow \quad r = -1, -2$$

   Therefore, $y_h(t) = C_1 e^{-t} + C_2 e^{-2t}$. Now use $y_p = Ae^{it}$, and substitute:
$$Ae^{it}(-1 + 3i + 2) = e^{it} \quad \Rightarrow \quad A = \frac{1}{1+3i}$$

   We want the real part of $Ae^{it}$. The full expansion is given below- Just pick out the real part for $y_p$:
$$Ae^{it} = \frac{1-3i}{10}(\cos(t) + i\sin(t))$$
   so $y_p = \frac{1}{10}\cos(t) + \frac{3}{10}\sin(t)$, or all together:
$$y(t) = C_1 e^{-t} + C_2 e^{-2t} + \frac{1}{10}\cos(t) + \frac{3}{10}\sin(t)$$

9. Consider $u'' + pu' + qu = \cos(\omega t)$. In the notes at the bottom of p. 4, we got that
$$\omega = \sqrt{\frac{2q - p^2}{2}}$$

   Thinking of $p$ as damping, if the damping is very very small, then approximately what value of $\omega$ will result in a very large amplitude response?

   SOLUTION: If the damping is very small, then the maximizer $\omega$ becomes very close to $\sqrt{q}$, which is what we would expect from no damping (and then resonance).

10. (Assigned as part of the quiz)

11. Consider $u'' + u' + 2u = \cos(\omega t)$. Find the value of $\omega$ that will maximize the amplitude of the response.

    NOTE: I don't want you to memorize the value of $\omega$. Rather, find the amplitude $R$, then differentiate to find where the derivative is zero. Remember our shortcut (dealing with $f(\omega)$).

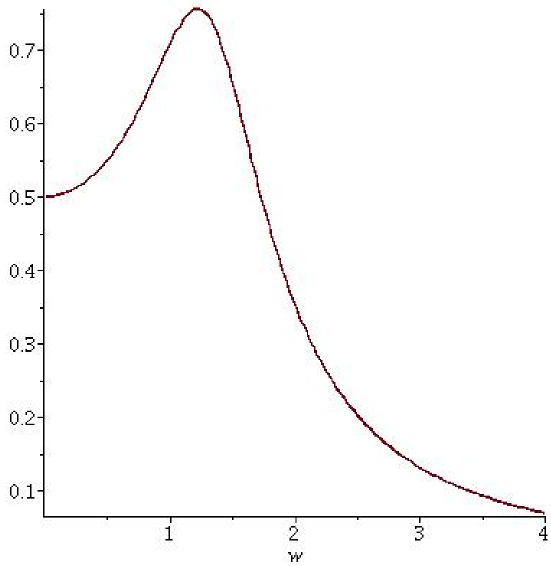
    SOLUTION: Let $y_p = Ae^{i\omega t}$, and substituting it into the DE:
$$Ae^{i\omega t}(-\omega^2 + i\omega + 2) = e^{i\omega t} \quad \Rightarrow \quad A = \frac{1}{(2-\omega^2) + i\omega}$$

The amplitude $R$ is therefore:

$$R = \frac{1}{|(2-\omega^2)+i\omega|} = \frac{1}{\sqrt{(2-\omega^2)^2+\omega^2}}$$

We looked at a shortcut for differentiating this and setting it to zero- That's the same as just differentiating $(2-\omega^2)^2+\omega^2$ and setting that to zero.

Doing that, we get $\omega = \sqrt{6}2 \approx 1.22$. For fun, we can can $R$ as a function of $\omega$ to see if we're accurate. Doing that, we get the figure below.




12. Pictured below are the graphs of several solutions to the differential equation:

$$y'' + py' + qy = \cos(\omega t)$$

Match the figure to the choice of parameters.

| Choice | $b$ | $c$ | $\omega$ |
|---|---|---|---|
| $(A)$ | 5 | 3 | 1 |
| $(B)$ | 0 | 2 | 1 |
| $(C)$ | 0 | 1 | 1 |
| $(D)$ | 2 | 1 | 3 |

SOLUTION: I wanted you to see that one of the graphs was BEATING (lower left, choice B), one was RESONANCE (upper left, choice C). To distinguish between the other two, I wanted you to estimate the periods and work it out that way. For the upper right graph, the period is approximately $2\pi$, so the forcing function would have $\omega = 1$ (choice A). The lower right function takes about $2\pi$ units to get 3 complete cycles, so $\omega = 3$ (and the choice is D)

13. Write the forced response to the ODE below as $R\cos(\omega t - \delta)$:

$$u'' + u' + 2u = \cos(3t)$$

SOLUTION: Using $y_p = Ae^{3it}$, we get

$$Ae^{3it}(-9+2i+2) = e^{3it} \quad \Rightarrow \quad A = \frac{1}{-7+3i}$$

The amplitude $R$ is then:

$$R = \frac{1}{|-7+2i|} = \frac{1}{\sqrt{58}} \qquad \delta = \tan^{-1}(-2/7) + \pi$$

NOTE: The actual, full $y_p$ can be computed as:

$$\frac{-7}{58}\cos(3t) + \frac{3}{58}\sin(3t)$$

then $R, \delta$ would be the same thing- But why would you want to go through all that work?

14. Suppose we can tune the value of $q$ rather than the value of $\omega$ in the differential equation (where $\omega = 3$):

$$u'' + u' + qu = \cos(3t)$$

Find the value of $q$ that will maximize the amplitude of the forced response.

SOLUTION: Go through the usual computation:

$$Ae^{3it}(-9+3i+q) = e^{3it} \quad \Rightarrow \quad A = \frac{1}{(q-9)+3i}$$

Therefore, the amplitude as a function of $q$ is:

$$R = \frac{1}{\sqrt{(q-9)^2+9}}$$

To find the $q$ that maximizes $R$, differentiate and set to zero. As before, we can use a shortcut:

$$\frac{d}{dq}(q-9)^2 + 9 = 0 \qquad \Rightarrow \quad q = 9$$