Exam 2 Summary

The exam will cover material from Section 3.1 to 3.7 except for 3.6 (Variation of Parameters). Here is a summary of that information.

Existence and Uniqueness:

Given the second order linear IVP,

$$y'' + p(t)y' + q(t)y = g(t), \qquad y(t_0) = y_0, \quad y'(t_0) = v_0$$

If there is an open interval I on which p, q, and g are continuous an contain t_0, then there exists a unique solution to the IVP, valid on I (and may contain the endpoints of I, if the functions are also continuous there).

Structure and Theory (Mostly 3.2)

The homogeneous equation.

The goal of the theory was to establish the structure of solutions to the second order IVP. We start with the **homogeneous** equation:

$$y'' + p(t)y' + q(t)y = 0, \qquad y(t_0) = y_0, \quad y'(t_0) = v_0$$

The "principle of superposition" says that if $y_1, y_2, \cdots, y_k$ are each a solution to the homogeneous equation, then so is the linear combination:

$$c_1y_1 + c_2y_2 + \cdots + c_ky_k$$

We further showed that for a second order linear homogeneous DE, we can write *any* solution using only y_1, y_2, as long as $W(y_1, y_2)(t_0) \neq 0$. In this case, y_1, y_2 form a **fundamental set of solutions**. This theory was all possible because the differential equation is **linear**.

The full, forced equation: $y'' + p(t)y' + q(t)y = g(t)$**,** $y(t_0) = y_0, y'(t_0) = v_0$

If a solution exists to the IVP, then it can always be expressed in the following form:

$$y(t) = C_1y_1(t) + C_2y_2(t) + y_p(t)$$

where y_1, y_2 form a fundamental set of solutions to the homogeneous equation, and $y_p(t)$ solves the non-homogeneous equation.

In fact, if we have: $y'' + p(t)y' + q(t)y = g_1(t) + g_2(t) + \ldots + g_n(t)$,, we can solve by splitting the problem up into smaller problems: Take y_{p_1} to be the solution to $y'' + p(t)y' + q(t)y = g_1(t)$, then y_{p_2} solves the DE with $g_2(t)$ on the RHS, and so on, until y_{p_n} solves $y'' + p(t)y' + q(t)y = g_n(t)$. Then the full solution is: $y(t) = C_1y_1 + C_2y_2 + y_{p_1} + y_{p_2} + \ldots + y_{p_n}$.

Abel's Theorem

If y_1, y_2 are solutions to $y'' + p(t)y' + q(t)y = 0$, then the Wronskian, $W(y_1, y_2)$, is either always zero or never zero on the interval for which the solutions are valid.

That is because the Wronskian may be computed as:

$$W(y_1, y_2)(t) = C\mathrm{e}^{-\int p(t)\,dt}$$

We can use Abel's Theorem to either compute the Wronskian, or it can help use solve the differential equation (if we have y_1, we can find y_2 by computing W two ways, explained below).

Finding the Homogeneous Solution

We had two distinct equations to solve-

$$ay'' + by' + cy = 0 \qquad \text{or} \qquad y'' + p(t)y' + q(t)y = 0$$

First we look at the case with constant coefficients, then we look at the more general case.

Constant Coefficients

To solve

$$ay'' + by' + cy = 0$$

we use the **ansatz** $y = \mathrm{e}^{rt}$. Then we form the associated **characteristic equation**:

$$ar^2 + br + c = 0 \qquad \Rightarrow \qquad r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

so that the solutions depend on the discriminant, $b^2 - 4ac$ in the following way:

- $b^2 - 4ac > 0 \Rightarrow$ two distinct real roots r_1, r_2. The general solution is:

  $$y_h(t) = c_1 \mathrm{e}^{r_1 t} + c_2 \mathrm{e}^{r_2 t}$$

  If $a, b, c > 0$ (as in the Spring-Mass model) we can further say that r_1, r_2 are negative. We would say that this system is OVERDAMPED.

- $b^2 - 4ac = 0 \Rightarrow$ one real root $r = -b/2a$. Then the general solution is:

  $$y_h(t) = \mathrm{e}^{-(b/2a)t} \left(C_1 + C_2 t\right)$$

  If $a, b, c > 0$ (as in the Spring-Mass model), the exponential term has a negative exponent. In this case (one real root), the system is CRITICALLY DAMPED.

- $b^2 - 4ac < 0 \Rightarrow$ two complex conjugate solutions, $r = \alpha \pm i\beta$. Then the solution is:

  $$y_h(t) = \mathrm{e}^{\alpha t} \left(C_1 \cos(\beta t) + C_2 \sin(\beta t)\right)$$

  If $a, b, c > 0$, then $\alpha = -(b/2a) < 0$. In the case of complex roots, the system is said to the UNDERDAMPED. If $\alpha = 0$ (this occurs when there is no damping), we get pure periodic motion, with period $2\pi/\beta$ or circular frequency β.

Solving the general homogeneous equation.

We had two methods for solving the more general equation:

$$y'' + p(t)y' + q(t)y = 0$$

but each method relied on already having one solution, $y_1(t)$. Given that situation, we can solve for y_2 (so that y_1, y_2 form a fundamental set), by one of two methods:

- By use of the Wronskian: There are two ways to compute this,

  - $W(y_1, y_2) = C\mathrm{e}^{-\int p(t)\,dt}$ (This is from Abel's Theorem)
  - $W(y_1, y_2) = y_1 y_2' - y_2 y_1'$

  Therefore, these are equal, and y_2 is the unknown: $y_1 y_2' - y_2 y_1' = C\mathrm{e}^{-\int p(t)\,dt}$

- Reduction of order, where $y_2 = v(t) y_1(t)$. Now substitute y_2 into the DE, and use the fact that y_1 solves the homogeneous equation, and the DE reduces to:

  $$y_1 v'' + (2y_1' + p y_1) v' = 0$$

Finding the particular solution.

Our two methods were: Method of Undetermined Coefficients and Variation of Parameters (but Variation of Parameters won't be on the exam).

Method of Undetermined Coefficients

This method is motivated by the observation that, a linear operator of the form $L(y) = ay'' + by' + cy$, acting on certain classes of functions, returns the same class. In summary, the table from the text:

if $g_i(t)$ is:	The ansatz y_{p_i} is:
$P_n(t)$	$t^s(a_0 + a_1 t + \ldots a_n t^n)$
$P_n(t)\mathrm{e}^{\alpha t}$	$t^s \mathrm{e}^{\alpha t}(a_0 + a_1 t + \ldots + a_n t^n)$
$P_n(t)\mathrm{e}^{\alpha t}\sin(\mu t)$ or $\cos(\mu t)$	$t^s \mathrm{e}^{\alpha t}\left((a_0 + a_1 t + \ldots + a_n t^n)\sin(\mu t)\right.$ $\left.+ (b_0 + b_1 t + \ldots + b_n t^n)\cos(\mu t)\right)$

The t^s term comes from an analysis of the homogeneous part of the solution. That is, multiply by t or t^2 so that no term of the ansatz is included as a term of the homogeneous solution.

Note that if we have periodic forcing, we can "complexify" the problem and make it easier to solve. For example,

$$ay'' + by' + cy = \cos(\omega t) \quad \to \quad ay'' + by' + cy = \cos(\omega t) + i\sin(\omega t) = \mathrm{e}^{i\omega t}$$

And then we use the guess for the exponential, $y_p = A\mathrm{e}^{i\omega t}$.

Analysis of the Oscillator Model (3.7-3.8)

Given

$$mu'' + \gamma u' + ku = F\cos(\omega t)$$

we should be able to determine the constants from a given setup for a spring-mass system. Once that's done, be able to analyze the spring-mass system in some particular cases:

1. Unforced (The homogeneous equation, $F = 0$)
   (a) No damping: Natural frequency is $\sqrt{k/m}$.
   (b) With damping: Underdamped, Critically Damped, Overdamped: These correspond to complex roots, equal roots, and two distinct real roots to the characteristic equation (respectively).
2. Periodic Forcing (Stop here for Exam 2).