Sample Questions (Chapter 3, Math 244)

1. True or False?

   (a) The characteristic equation for $y'' + y' + y = 1$ is $r^2 + r + 1 = 1$

   (b) The characteristic equation for $y'' + xy' + e^x y = 0$ is $r^2 + xr + e^x = 0$

   (c) The function $y = 0$ is always a solution to a second order linear homogeneous differential equation.

   (d) In using the Method of Undetermined Coefficients, the ansatz $y_p = (Ax^2 + Bx + C)(D\sin(x) + E\cos(x))$ is equivalent to
   $$y_p = (Ax^2 + Bx + C)\sin(x) + (Dx^2 + Ex + F)\cos(x)$$

   (e) Consider the function:
   $$y(t) = \cos(t) - \sin(t)$$
   Then amplitude is 1, the period is 1 and the phase shift is 0.

2. Find values of a for which **any** solution to:
   $$y'' + 10y' + ay = 0$$
   will tend to zero (that is, $\lim_{t\to 0} y(t) = 0$.

3.
   - Compute the Wronskian between $f(x) = \cos(x)$ and $g(x) = 1$.
   - Can these be two solutions to a second order linear homogeneous differential equation? Be specific. (Hint: Abel's Theorem)

4. Construct the operator associated with the differential equation: $y' = y^2 - 4$. Is the operator linear? Show that your answer is true by using the definition of a linear operator.

5. (i) Solve: $u'' + \omega_0^2 u = F_0\cos(\omega t), \quad u(0) = 0 \quad u'(0) = 0$ if $\omega \neq \omega_0$ using the Method of Undetermined Coefficients. (ii) Repeat the problem if $\omega = \omega_0$.

6. Given that $y_1 = \frac{1}{t}$ solves the differential equation:
   $$t^2 y'' - 2y = 0$$
   Find a fundamental set of solutions using Abel's Theorem.

7. Suppose a mass of 0.01 kg is suspended from a spring, and the damping factor is $\gamma = 0.05$. If there is no external forcing, then what would the spring constant have to be in order for the system to *critically damped? underdamped?*

8. Give the full solution, using any method(s). If there is an initial condition, solve the initial value problem.

   (a) $y'' + 2y' + 2y = 0$.

   (b) $u'' + u = 3t + 4$, $u(0) = 0, u'(0) = 0$

   (c) $y'' + 4y' + 4y = e^{-2t}$

   (d) $y'' - 2y' + y = te^t + 4$, $y(0) = 1$, $y'(0) = 1$.

   (e) $y'' + y' - 2y = 4t$

   (f) $4y'' - 4y' + y = 16e^t$

9. For each problem below, write the *form* of $y_p(t)$ using the Method of Undetermined Coefficients, but do NOT solve for the coefficients.

   (a) $y'' + 2y' + 2y = te^{-t}(1 + \sin(t))$

   (b) $y'' + 2y' = 2t^4 + \sin(2t)$

   (c) $y'' + 4y = t^2\sin(2t)$

10. Solve for y_p only by complexifying the problem first:

(a) $y'' + 2y' + 3y = \cos(2t)$ (b) $y'' - y' + 3y = \cos(3t)$ (c) $y'' + 9y = \sin(2t)$

11. Rewrite the expression in the form $a + ib$: (i) 2^{i-1} (ii) $e^{(3-2i)t}$ (iii) $e^{i\pi}$

12. Write $a + ib$ in polar form: (i) $-1 - \sqrt{3}i$ (ii) $3i$ (iii) -4 (iv) $\sqrt{3} - i$

13. Write each function as $R\cos(\omega t - \delta)$ for an appropriate R, δ.

    (a) $f(t) = \cos(3t) - \sqrt{3}\sin(3t)$ (b) $h(t) = -\sqrt{3}\cos(3t) + \sin(3t)$ (c) $g(t) = \cos(t) + \sin(t)$

14. Find a second order linear differential equation with constant coefficients whose general solution is given by:
$$y(t) = C_1 + C_2 e^{-t} + \frac{1}{2}t^2 - t$$

15. Determine the longest interval for which the IVP is certain to have a unique solution (Do not solve the IVP):
$$t(t-4)y'' + 3ty' + 4y = 2 \qquad y(3) = 0 \quad y'(3) = -1$$

16. Let $L(y) = ay'' + by' + cy$ for some value(s) of a, b, c.
    If $L(3e^{2t}) = -9e^{2t}$ and $L(t^2 + 3t) = 5t^2 + 3t - 16$, what is the particular solution to:
$$L(y) = -10t^2 - 6t + 32 + e^{2t}$$

17. Compute the Wronskian of two solutions of the given DE without solving it:
$$x^2 y'' + xy' + (x^2 - \alpha^2)y = 0$$

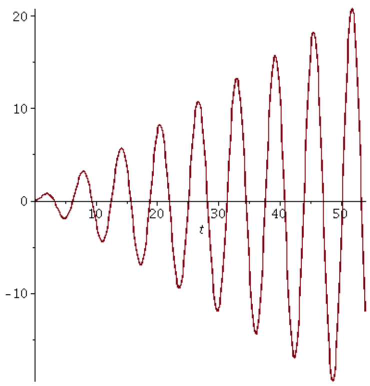
18. If $y'' - y' - 6y = 0$, with $y(0) = 1$ and $y'(0) = \alpha$, determine the value(s) of α so that the solution tends to zero as $t \to \infty$.

19. A mass of 0.5 kg stretches a spring an additional 0.05 meters to get to equilibrium. (i) Find the spring constant. (ii) Does a stiff spring have a large spring constant or a small spring constant (explain).

20. A mass of $\frac{1}{2}$ kg is attached to a spring with spring constant 2 (kg/sec^2). The spring is pulled down an additional 1 meter then released. Find the equation of motion if the damping constant is $c = 2$ as well:

21. Match the solution curve to its IVP (There is one DE with no graph, and one graph with no DE- You should not try to completely solve each DE).

    (a) $5y'' + y' + 5y = 0$, $y(0) = 10$, $y'(0) = 0$
    (b) $y'' + 5y' + y = 0$, $y(0) = 10$, $y'(0) = 0$
    (c) $y'' + y' + \frac{5}{4}y = 0$, $y(0) = 10$, $y'(0) = 0$
    (d) $5y'' + 5y = 4\cos(t)$, $y(0) = 0$, $y'(0) = 0$
    (e) $y'' + \frac{1}{2}y' + 2y = 10$, $y(0) = 0$, $y'(0) = 0$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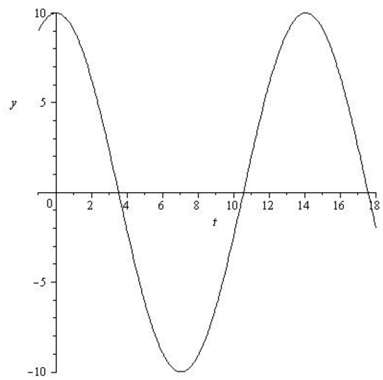
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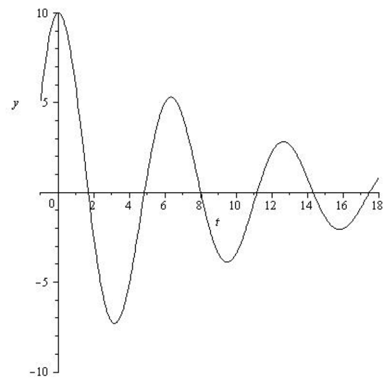
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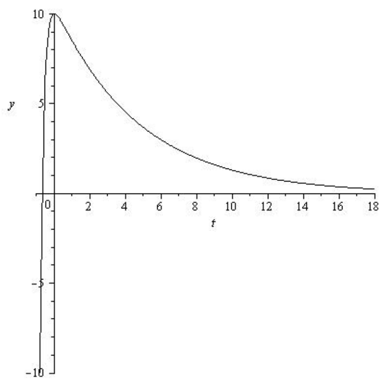
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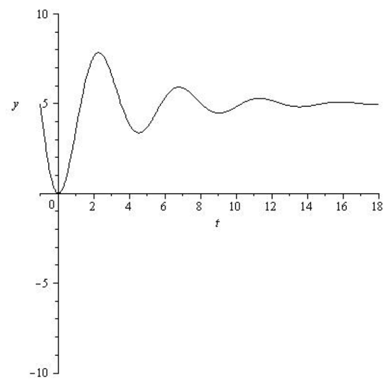
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