Sample Question Solutions (Chapter 3, Math 244)

1. True or False?

   (a) The characteristic equation for $y'' + y' + y = 1$ is $r^2 + r + 1 = 1$
   SOLUTION: False. The characteristic equation is for the homogeneous equation, $r^2 + r + 1 = 0$

   (b) The characteristic equation for $y'' + xy' + e^x y = 0$ is $r^2 + xr + e^x = 0$
   SOLUTION: False. The characteristic equation was defined only for DEs with constant coefficients, since our ansatz depended on constant coefficients.

   (c) The function $y = 0$ is always a solution to a second order linear homogeneous differential equation.
   SOLUTION: True. It is true generally- If L is a linear operator, then $L(0) = 0$.

   (d) In using the Method of Undetermined Coefficients, the ansatz $y_p = (Ax^2 + Bx + C)(D\sin(x) + E\cos(x))$ is equivalent to

   $$y_p = (Ax^2 + Bx + C)\sin(x) + (Dx^2 + Ex + F)\cos(x)$$

   SOLUTION: False- We have to be able to choose the coefficients for each polynomial (for the sine and cosine) independently of each other. In the form:

   $$(Ax^2 + Bx + C)(D\sin(x) + E\cos(x))$$

   the polynomials for the sine and cosine are constant multiples of each other, which may not necessarily hold true. That's why we need one polynomial for the sine, and one for the cosine (so the second guess is the one to use).

   (e) Consider the function:

   $$y(t) = \cos(t) - \sin(t)$$

   Then amplitude is 1, the period is 1 and the phase shift is 0.
   SOLUTION: False. For this question to make sense, we have to first write the function as $R\cos(\omega t - \delta)$. In this case, the amplitude is R:

   $$R = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

   The period is 2π (the circular frequency, or natural frequency, is 1), and the phase shift δ is:

   $$\tan(\delta) = -1 \quad \Rightarrow \quad \delta = -\frac{\pi}{4}$$

2. Find values of a for which **any** solution to:

   $$y'' + 10y' + ay = 0$$

   will tend to zero (that is, $\lim_{t\to\infty} y(t) = 0$.

SOLUTION: Use the characteristic equation and check the 3 cases (for the discriminant). That is,

$$r^2 + 10r + a = 0 \quad \Rightarrow \quad r = \frac{-10 \pm \sqrt{100 - 4a}}{2}$$

We check some special cases:

- If $100 - 4a = 0$ (or $a = 25$), we get a double root, $r = -5, -5$, or $y_h = \mathrm{e}^{-5t}(C_1 + C_2 t)$, and all solutions tend to zero.
- If the roots are complex, then we can write $r = -5 \pm \beta i$, and we get

  $$y_h = \mathrm{e}^{-5t}(C_1 \cos(\beta t) + C_2 \sin(\beta t))$$

  and again, this will tend to zero for any choice of C_1, C_2.
- In the case that $a < 25$, we have to be a bit careful. While it is true that both roots will be *real*, we also want them to both be *negative* for all solutions to tend to zero.
  - When will they both be negative? If $100 - 4a < 100$ (or $\sqrt{100 - 4a} < 10$). This happens as long as $a > 0$.
  - If $a = 0$, the roots will be $r = -10, 0$, and $y_h = C_1 \mathrm{e}^{-10t} + C_2$- Therefore, I could choose $C_1 = 0$ and $C_2 \neq 0$, and my solution will not go to zero.
  - If $a < 0$, the roots will be mixed in sign (one positive, one negative), so the solutions will not all tend to zero.

CONCLUSION: If $a > 0$, all solutions to the homogeneous will tend to zero.

Side Remark: You might recall that we showed that, if $a, b, c > 0$, then all solutions to $ay'' + by' + cy = 0$ go to zero as $t \to \infty$, no matter the initial conditions.

3.
- Compute the Wronskian between $f(x) = \cos(x)$ and $g(x) = 1$.
  SOLUTION: $W(\cos(x), 1) = \sin(x)$
- Can these be two solutions to a second order linear homogeneous differential equation? Be specific. (Hint: Abel's Theorem)
  SOLUTION: Abel's Theorem tells us that the Wronskian between two solutions to a second order linear homogeneous DE will either be identically zero or never zero on the interval on which the solution(s) are defined.

  Therefore, as long as the interval for the solutions do not contain a multiple of π (for example, $(0, \pi)$, $(\pi, 2\pi)$, etc), then it is possible for the Wronskian for two solutions to be $\sin(x)$.

4. Construct the operator associated with the differential equation: $y' = y^2 - 4$. Is the operator linear? Show that your answer is true by using the definition of a linear operator.

SOLUTION: The operator is found by getting all terms in y to one side of the equation, everything else on the other. In this case, we have:

$$L(y) = y' - y^2$$

This is not a linear operator. We can check using the definition:

$$L(cy) = cy' - c^2y^2 \neq cL(y)$$

Furthermore,

$$L(y_1 + y_2) = (y_1' + y_2') - (y_1 + y_2)^2 \neq L(y_1) + L(y_2)$$

5. Solving the undamped, periodically forced equation two ways:

   (a) Solve: $u'' + \omega_0^2 u = F_0 \cos(\omega t), \quad u(0) = 0 \quad u'(0) = 0$ if $\omega \neq \omega_0$ using the Method of Undetermined Coefficients.

   SOLUTION: The characteristic equation is: $r^2 + \omega_0^2 = 0$, or $r = \pm\omega_0 i$. Therefore,

$$u_h = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t)$$

   Using the Method of Undetermined Coefficients, $u_p = A\mathrm{e}^{i\omega t}$. Substitution into the DE:

$$-\omega^2 A\mathrm{e}^{i\omega t} + \omega_0^2 A\mathrm{e}^{i\omega t} = F_0\mathrm{e}^{i\omega t} \quad \Rightarrow \quad A = \frac{F_0}{\omega_0^2 - \omega^2}$$

   This expression is real, so the particular solution is this constant times the cosine. Putting it all together so far, the general solution is

$$u(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) + \frac{F_0}{\omega_0^2 - \omega^2} \cos(\omega t)$$

   Put in the initial conditions $u(0) = 0$ and $u'(0) = 0$ to see that $C_1 = -\frac{F_0}{\omega_0^2 - \omega^2}$ and $C_2 = 0$ so that

$$u(t) = \frac{F_0(\cos(\omega t) - \cos(\omega_0 t))}{\omega_0^2 - \omega^2}$$

   (b) Compute the solution to: $u'' + \omega_0^2 u = F_0 \cos(\omega_0 t) \qquad u(0) = 0 \quad u'(0) = 0$ with Method of Undetermined Coefficients

   SOLUTION: Let $u_p = At\mathrm{e}^{i\omega_0 t}$. Then

$$u_p' = A\mathrm{e}^{i\omega_0 t} + i\omega_0 At\mathrm{e}^{i\omega_0 t} \qquad u_p'' = 2i\omega_0 A\mathrm{e}^{i\omega_0 t} - \omega_0^2 At\mathrm{e}^{i\omega_0 t}$$

   Now,

$$u_p'' + \omega_0^2 u_p = 2i\omega A\mathrm{e}^{i\omega t} = F_0\mathrm{e}^{i\omega_0 t} \quad \Rightarrow \quad A = \frac{F_0}{2i\omega_0} = -\frac{F_0}{2\omega_0}\, i$$

We want the real part of $Ate^{i\omega_0 t}$:

$$Ate^{i\omega_0 t} = -\frac{F_0}{2\omega_0}\, it(\cos(\omega_0\, t) + i\sin(\omega_0 t))$$

which we see is:

$$u_p = \frac{F_0}{2\omega_0} t\, \sin(\omega_0\, t)$$

Putting it all together,

$$u(t) = C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t) + \frac{F_0}{2\omega_0} t\sin(\omega_0 t)$$

Now, $u(0) = 0$ means that $C_1 = 0$. Differentiating for the second IC,

$$u' = \omega_0 C_2\cos(\omega_0\, t) + \frac{F_0}{2\omega_0}\sin(\omega_0 t) + \frac{F_0\omega_0}{2\omega_0} t\cos(\omega_0 t)$$

In this case, $C_2 = 0$ as well, so the particular solution is the full solution.

6. Given that $y_1 = \frac{1}{t}$ solves the differential equation:

$$t^2 y'' - 2y = 0$$

Find a fundamental set of solutions using Abel's Theorem:

SOLUTION: First, rewrite the differential equation in standard form:

$$y'' - \frac{2}{t^2} y = 0$$

Then $p(t) = 0$ and $W(y_1, y_2) = Ce^0 = C$. On the other hand, the Wronskian is:

$$W(y_1, y_2) = \frac{1}{t} y_2' + \frac{1}{t^2} y_2$$

Put these together:

$$\frac{1}{t} y_2' + \frac{1}{t^2} y_2 = C \quad y_2' + \frac{1}{t} y_2 = Ct$$

The integrating factor is t,

$$(ty_2)' = Ct^2 \quad \Rightarrow \quad ty_2 = C_1 t^3 + C_2 \quad \Rightarrow \quad C_1 t^2 + \frac{C_2}{t}$$

Notice that we have *both* parts of the homogeneous solution, $y_1 = \frac{1}{t}$ and $y_2 = t^2$.

7. Suppose a mass of 0.01 kg is suspended from a spring, and the damping factor is $\gamma = 0.05$. If there is no external forcing, then what would the spring constant have to be in order for the system to *critically damped? underdamped?*

SOLUTION: We can find the differential equation:

$$0.01u'' + 0.05u' + ku = 0 \quad \Rightarrow \quad u'' + 5u' + 100ku = 0$$

Then the system is *critically damped* if the discriminant (from the quadratic formula) is zero:

$$b^2 - 4ac = 25 - 4 \cdot 100k = 0 \quad \Rightarrow \quad k = \frac{25}{400} = \frac{1}{16}$$

The system is *underdamped* if the discriminant is negative:

$$25 - 400k < 0 \quad \Rightarrow \quad k > \frac{1}{16}$$

8. Give the full solution, using any method(s). If there is an initial condition, solve the initial value problem.

(a) $y'' + 2y' + 2y = 0$
SOLUTION: The char eqn is given by $r^2 + 2r + 2 = 0$. Since $b^2 - 4ac < 0$, we complete the square:

$$r^2 + 2r = -2 \quad \rightarrow \quad r^2 + 2r + 1 = -1 \quad \rightarrow \quad (r+1)^2 = -1 \quad \rightarrow \quad r = -1 \pm i$$

Therefore, $y(t) = \mathrm{e}^{-t}(C_1 \cos(t) + C_2 \sin(t))$

(b) $u'' + u = 3t + 4$, zero ICs.
SOLUTION: We see that $y_h(t) = C_1 \cos(t) + C_2 \sin(t)$, and $y_p(t) = At + B$ (by Method of Undet Coeffs). Substituting,

$$At + B = 3t + 4 \quad \Rightarrow \quad A = 3, B = 4$$

The solution thus far is $C_1 \cos(t) + C_2 \sin(t) + 3t + 4$. Using the inital conditions,

$$u(0) = 0 \quad \Rightarrow \quad 0 = C_1 + 4 \quad \Rightarrow \quad C_1 = -4$$

$$u'(0) = 0 \quad \Rightarrow \quad 0 = C_2 + 3$$

Therefore,

$$u(t) = -4\cos(t) - 3\sin(t) + 3t + 4$$

(c) $y'' + 4y' + 4y = \mathrm{e}^{-2t}$
SOLUTUION: We see that

$$y_h(t) = C_1\mathrm{e}^{-2t} + C_2 t\mathrm{e}^{-2t}$$

so that (in putting down our guess, multiply by t^2):

$$y_p = At^2 \mathrm{e}^{-2t}$$

Substituting this into the DE, we should find that $A = 1/2$, so that the full solution is

$$y(t) = C_1 \mathrm{e}^{-2t} + C_2 t\mathrm{e}^{-2t} + \frac{1}{2} t^2 \mathrm{e}^{-2t}$$

(d) $y'' - 2y' + y = t\mathrm{e}^t + 4$, $y(0) = 1$, $y'(0) = 1$.
With the Method of Undetermined Coefficients, we first get the homogeneous part of the solution,

$$y_h(t) = \mathrm{e}^t (C_1 + C_2 t)$$

Now we construct our ansatz (Multiplied by t after comparing to y_h):

$$g_1 = t\mathrm{e}^t \quad \Rightarrow \quad y_{p_1} = (At + B)\mathrm{e}^t \cdot t^2$$

Substitute this into the differential equation to solve for A, B:

$$y_{p_1} = (At^3 + Bt^2)\mathrm{e}^t \qquad y'_{p_1} = (At^3 + (3A + B)t^2 + 2Bt)\mathrm{e}^t$$

$$y''_{p_1} = (At^3 + (6A + B)t^2 + (6A + 4B)t + 2B)\mathrm{e}^t$$

Forming $y''_{p_1} - 2y'_{p_1} + y_{p_1} = t\mathrm{e}^t$, we should see that $A = \frac{1}{6}$ and $B = 0$, so that $y_{p_1} = \frac{1}{6} t^3 \mathrm{e}^t$.
The next one is a lot easier! $y_{p_2} = A$, so $A = 4$, and:

$$y(t) = \mathrm{e}^t (C_1 + C_2 t) + \frac{1}{6} t^3 \mathrm{e}^t + 4$$

with $y(0) = 1$, $C_1 = -3$. Solving for C_2 by differentiating should give $C_2 = 4$. The full solution:

$$y(t) = \mathrm{e}^t \left(\frac{1}{6} t^3 + 4t - 3 \right) + 4$$

(e) $y'' + y' - 2y = 4t$.
Characteristic equation: $r^2 + r - 2 = 0$, or $(r + 2)(r - 1) = 0$, so $r = 1, -4$. Therefore,

$$y_h(t) = C_1 \mathrm{e}^t + C_2 \mathrm{e}^{-4t}$$

And we'll guess that $y_p = At + B$, so $y'_p = A$ and $y''_p = 0$, so

$$A - 2(At + B) = 4t \qquad \begin{aligned} -2A &= 4 \\ A - 2B &= 0 \end{aligned} \qquad A = -2, B = -1$$

Therefore, $y(t) = C_1 \mathrm{e}^t + C_2 \mathrm{e}^{-4t} - 2t - 1$

(f) $4y'' - 4y' + y = 16e^t$

SOLUTION: The characteristic equation:

$$4r^2 - 4r + 1 = 0 \quad \to \quad 4(r^2 - r) = -1 \quad \to \quad 4\left(r^2 - r + \frac{1}{4}\right) = 0 \quad \to \quad r = \frac{1}{2}, \frac{1}{2}$$

For $y_p = Ae^t$, then $4Ae^t - 4Ae^t + Ae^t = 16e^t$, so $A = 16$ and the full solution

$$y(t) = e^{t/2}(C_1 + C_2 t) + 16e^t$$

9. For each problem below, write the *form* of $y_p(t)$ using the Method of Undetermined Coefficients, but do NOT solve for the coefficients.

(a) $y'' + 2y' + 2y = te^{-t}(1 + \sin(t))$

SOLUTION: The roots to the characteristic equation are: $r^2 + 2r + 2 = 0$, or $r^2 + 2r + 1 = -1$, or $r = -1 \pm i$. That means the solution to the characteristic equation uses functions $e^{-t}\cos(t)$ and $e^{-t}\sin(t)$.

For the particular solution, let's multiply things out: $F(t) = te^{-t} + te^{-t}\sin(t) = g_1(t) + g_2(t)$.

- For $g_1(t) = te^{-t}$, we guess: $y_{p_1}(t) = (At + B)e^{-t}$
- For $g_2(t) = te^{-t}\sin(t)$, we guess: $y_{p_2}(t) = e^{-t}[(At+B)\cos(t) + (Ct+D)\sin(t)]$. We have to multiply that by t because they overlap with our homogeneous solutions:

$$y_{p_2}(t) = te^{-t}[(At + B)\cos(t) + (Ct + D)\sin(t)]$$

(b) $y'' + 2y' = 2t^4 + \sin(2t)$

SOLUTION: For the homogeneous equation, we have $r^2 + 2r = 0$, or $r = 0, -2$. Therefore, $y_1 = 1$ and $y_2 = e^{-2t}$. Break apart the forcing function as we did last time:

- For $g_1(t) = 2t^4$, we guess $y_{p_1}(t) = At^4 + Bt^3 + Ct^2 + Dt + E$. Notice that $y_1 = 1$ is a constant term, and here E is a constant term, so we have to multiply our guess by t:

$$y_{p_1}(t) = t(At^4 + Bt^3 + Ct^2 + Dt + E)$$

- For $g_2(t) = \sin(2t)$, we guess $y_{p_2}(t) = A\cos(2t) + B\sin(2t)$.

(c) $y'' + 4y = t^2\sin(2t)$ SOLUTION: For the homogeneous equation, we have $r^2 + 4 = 0$, or $r = \pm 2i$. Therefore, $y_1 = \cos(2t)$ and $y_2 = \sin(2t)$. Now for the particular solution, we have a polynomial of degree 2 times the sine, so we guess:

$$y_p = (At^2 + Bt + C)\cos(2t) + (Ct^2 + Dt + E)\sin(2t)$$

But this includes the homogeneous solution, so multiply it by t:

$$y_p = t[(At^2 + Bt + C)\cos(2t) + (Ct^2 + Dt + E)\sin(2t)]$$

10. Solve for y_p only by complexifying the problem first:

(a) $y'' + 2y' + 3y = \cos(2t)$
SOLUTION: $y'' + 2y' + 3y = \mathrm{e}^{2it}$, so $y_p = A\mathrm{e}^{2it}$ and if we substitute this into the DE, we can factor out $A\mathrm{e}^{2it}$ and get

$$A\mathrm{e}^{2it}(-4 + 2(2i) + 3) = \mathrm{e}^{2it} \quad \Rightarrow \quad A = \frac{1}{-1 + 4i}$$

The particular solution is then the real part of $A\mathrm{e}^{2it}$, which we found to be:

$$y_p = \frac{-1}{1^2 + 4^2}\cos(2t) + \frac{4}{1^2 + 4^2}\sin(2t) = \frac{-1}{17}\cos(2t) + \frac{4}{17}\sin(2t)$$

(b) $y'' - y' + 3y = \cos(3t)$
SOLUTION: Same technique as before, with $y_p = A\mathrm{e}^{3it}$

$$A\mathrm{e}^{3it}(-9 - (3i) + 3) = \mathrm{e}^{3it} \quad \Rightarrow \quad A = \frac{1}{-6 - 3i}$$

The particular part of the solution is therefore:

$$y_p(t) = \frac{-6}{6^2 + 3^2}\cos(3t) + \frac{-3}{6^2 + 3^2}\sin(3t) = -\frac{6}{45}\cos(3t) - \frac{3}{45}\sin(3t)$$

(c) $y'' + 9y = \sin(2t)$
SOLUTION: We start the same as before, but then we take the imaginary part of $A\mathrm{e}^{2it}$.

$$A\mathrm{e}^{2it}(-4 + 9) = \mathrm{e}^{2it} \quad \Rightarrow \quad A = \frac{1}{-5}$$

Now, $A\mathrm{e}^{2it}$ is easy to compute since A is a real number. We want the imaginary part:

$$A\mathrm{e}^{2it} = \frac{-1}{5}(\cos(2t) + i\sin(2t)) \quad \Rightarrow \quad y_p(t) = -\frac{1}{5}\sin(2t)$$

11. Rewrite the expression in the form $a + ib$: (i) 2^{i-1} (ii) $\mathrm{e}^{(3-2i)t}$ (iii) $\mathrm{e}^{i\pi}$

NOTE for the SOLUTION: Remember that for any non-negative number A, we can write $A = \mathrm{e}^{\ln(A)}$.

- $2^{i-1} = \mathrm{e}^{ln(2^{i-1})} = \mathrm{e}^{(i-1)\ln(2)} = \mathrm{e}^{-\ln(2)}\mathrm{e}^{i\ln(2)} = \frac{1}{2}\left(\cos(\ln(2)) + i\sin(\ln(2))\right)$
- $\mathrm{e}^{(3-2i)t} = \mathrm{e}^{3t}\mathrm{e}^{-2ti} = \mathrm{e}^{3t}\left(\cos(-2t) + i\sin(-2t)\right) = \mathrm{e}^{3t}\left(\cos(2t) - i\sin(2t)\right)$
  (Recall that cosine is an even function, sine is an odd function).
- $\mathrm{e}^{i\pi} = \cos(\pi) + i\sin(\pi) = -1$

12. Write $a + ib$ in polar form: (i) $-1 - \sqrt{3}i$ (ii) $3i$ (iii) -4 (iv) $\sqrt{3} - i$

SOLUTIONS:

(i) $r = \sqrt{1+3} = 2$, $\theta = -2\pi/3$ (look at its graph, use 30-60-90 triangle):

$$-1 - \sqrt{3}i = 2e^{-\frac{2\pi}{3}i}$$

(ii) $3i = 3e^{\frac{\pi}{2}i}$

(iii) $-4 = 4e^{\pi i}$

(iv) $\sqrt{3} - i = 2e^{-\frac{\pi}{6}i}$

13. Write each function as $R\cos(\omega t - \delta)$ for an appropriate R, δ.

(a) $f(t) = \cos(3t) - \sqrt{3}\sin(3t)$
SOLUTION: $R = \sqrt{1^2 + (\sqrt{3}^2} = 2$. For the angle, think about the point being $1-\sqrt{3}i$, so that the angle is in the 4th quadrant (we won't need to add π). Further, we might recognize that the angle will be coming from the 30-60-90 triangle (with side lengths 1-2-$\sqrt{3}$), so sketching that triangle will give you:

$$\delta = \tan^{-1}(-\sqrt{3}/1) = -\pi/3$$

Therefore,

$$\cos(3t) - \sqrt{3}\sin(3t) = 2\cos(3t + \pi/3)$$

(b) $h(t) = -\sqrt{3}\cos(3t) + \sin(3t)$
Same technique as before, but note that $-\sqrt{3} + i$ is in quadrant II, so we need to add π to the inverse tangent.

$$\delta = \tan^{-1}(-1/\sqrt{3}) + \pi = -\pi/6 + \pi = 5\pi/6$$

Now,

$$-\sqrt{3}\cos(3t) + \sin(3t) = 2\cos(3t - 5\pi/6)$$

(c) $g(t) = \cos(t) + \sin(t)$

$$\cos(t) + \sin(t) = \sqrt{2}\cos(t - \pi/4)$$

14. Find a second order linear differential equation with constant coefficients whose general solution is given by:

$$y(t) = C_1 + C_2e^{-t} + \frac{1}{2}t^2 - t$$

SOLUTION: Work backwards from the characteristic equation to build the homogeneous DE (then figure out the rest):

The roots to the characteristic equation are $r = 0$ and $r = -1$. The characteristic equation must be $r(r+1) = 0$ (or a constant multiple of that). Therefore, the differential equation is:

$$y'' + y' = 0$$

For $y_p = \frac{1}{2}t^2 - t$ to be the particular solution,

$$y_p'' + y_p' = (1) + (t-1) = t$$

so the full differential equation must be:

$$y'' + y' = t$$

15. Determine the longest interval for which the IVP is certain to have a unique solution (Do not solve the IVP):

$$t(t-4)y'' + 3ty' + 4y = 2 \qquad y(3) = 0 \quad y'(3) = -1$$

SOLUTION: Write in standard form first-

$$y'' + \frac{3}{t-4}y' + \frac{4}{t(t-4)}y = \frac{2}{t(t-4)}$$

The coefficient functions are all continuous on each of three intervals:

$$(-\infty, 0), (0, 4) \text{ and } (4, \infty)$$

Since the initial time is 3, we choose the middle interval, $(0, 4)$.

16. Let $L(y) = ay'' + by' + cy$ for some value(s) of a, b, c.

If $L(3e^{2t}) = -9e^{2t}$ and $L(t^2 + 3t) = 5t^2 + 3t - 16$, what is the particular solution to:

$$L(y) = -10t^2 - 6t + 32 + e^{2t}$$

SOLUTION: This purpose of this question is to see if we can use the properties of linearity to get at the answer.

We see that: $L(3e^{2t}) = -9e^{2t}$, or $L(e^{2t}) = -3e^{2t}$ so:

$$L\left(-\frac{1}{3}e^{2t}\right) = e^{2t}$$

And for the second part,

$$L(t^2 + 3t) = 5t^2 + 3t - 16 \quad \Rightarrow \quad L((-2)(t^2 + 3t)) = -10t^2 + 6t - 32$$

The particular solution is therefore:

$$y_p(t) = -2(t^2 + 3t) - \frac{1}{3}e^{2t}$$

since we have shown that

$$L\left(-2(t^2 + 3t) - \frac{1}{3}e^{2t}\right) = -10t^2 + 6t - 32 + e^{2t}$$

17. Compute the Wronskian of two solutions of the given DE without solving it:

$$x^2y'' + xy' + (x^2 - \alpha^2)y = 0$$

Using Abel's Theorem (and writing the DE in standard form first):

$$y'' + \frac{1}{x}y' + \frac{x^2 - \alpha^2}{x^2}y = 0$$

Therefore,

$$W(y_1, y_2) = C\mathrm{e}^{-\int \frac{1}{x}\,dx} = \frac{C}{x}$$

18. If $y'' - y' - 6y = 0$, with $y(0) = 1$ and $y'(0) = \alpha$, determine the value(s) of α so that the solution tends to zero as $t \to \infty$.

SOLUTION: Solving as usual gives:

$$y(t) = \left(\frac{3-\alpha}{5}\right)\mathrm{e}^{-2t} + \left(\frac{\alpha+2}{5}\right)\mathrm{e}^{3t}$$

so to make sure the solutions tend to zero, $\alpha = -2$ (to zero out the second term).

19. A mass of 0.5 kg stretches a spring an additional 0.05 meters to get to equilibrium. (i) Find the spring constant. (ii) Does a stiff spring have a large spring constant or a small spring constant (explain).

SOLUTION:

We use Hooke's Law at equilibrium: $mg - kL = 0$, or

$$k = \frac{mg}{L} = \frac{4.9}{0.05} = 98$$

For the second part, a stiff spring will not stretch, so L will be small (and k would therefore be large), and a spring that is not stiff will stretch a great deal (so that k will be smaller).

20. A mass of $\frac{1}{2}$ kg is attached to a spring with spring constant 2 (kg/sec^2). The spring is pulled down an additional 1 meter then released. Find the equation of motion if the damping constant is $c = 2$ as well:

SOLUTION: Just substitute in the values

$$\frac{1}{2}u'' + 2u' + 2u = 0$$

Pulling down the spring and releasing: $u(0) = 1$, $u'(0) = 0$ (Down is positive)

21. Match the solution curve to its IVP (There is one DE with no graph, and one graph with no DE- You should not try to completely solve each DE).

   (a) $5y'' + y' + 5y = 0$, $y(0) = 10$, $y'(0) = 0$ (Complex roots, solutions go to zero)
   (b) $y'' + 5y' + y = 0$, $y(0) = 10$, $y'(0) = 0$ (Exponentials, solutions go to zero)
   (c) $y'' + y' + \frac{5}{4}y = 0$, $y(0) = 10$, $y'(0) = 0$
   (d) $5y'' + 5y = 4\cos(t)$, $y(0) = 0$, $y'(0) = 0$
   (e) $y'' + \frac{1}{2}y' + 2y = 10$, $y(0) = 0$, $y'(0) = 0$

   SOLUTION:

   There are three graphs going through $(10, 0)$ and two going through $(0, 0)$, so that should help you classify them. Of the three going through $(10, 0)$, we see one graph that is purely periodic (no damping), one that oscillates and decays to zero (underdamped), and one that goes fairly quickly to zero (probably overdamped). Equations (a) and (c) are both candidates for the underdamped graph- I meant for them to be more different (Sorry!). You can see from the graph that the "pseudoperiod" is 2π, which would correspond to (c) (For this exam, I won't go this in depth). The first graph will correspond to equation (4), where the particular part of the solution would have to be $y_p = t(A\cos(t) + B\sin(t)$, so the solution "blows up". The second graph (purely periodic) is not used by any of the equations.