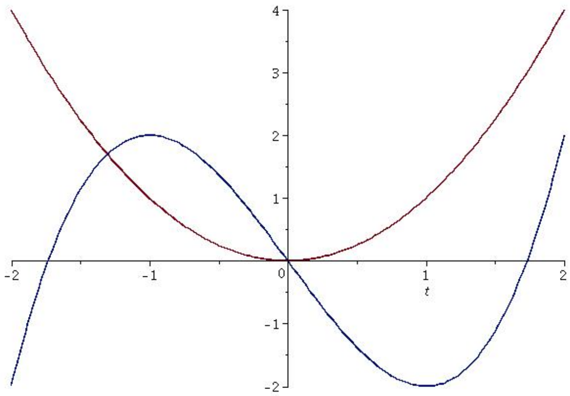
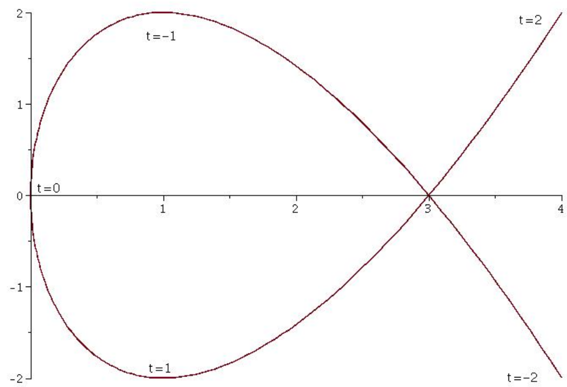
Solutions- Homework set to replace 7.1

1. If $x(t)$ and $y(t)$ are plotted below (versus time), plot the graph in the xy-plane. (Hint: You might think about some specific points in the $xy-$plane). Remember that at $t=-2$, you're starting all the way to the left, and at $t=2$, you're all the way to the right (in the $t-$plane). You may assume that the $x-$coordinate is the parabola.





Hint: If you're having trouble, try plotting specific points and transfer those points onto the xy plane.

2. Exercise 22, p 363 (Section 7.1)

Construct the DE's by looking at "Rate In - Rate Out", and make sure your units are matching up. Define $Q_1(t)$ and $Q_2(t)$ to be the ounces of salt in Tanks 1 and 2 (respectively). Before we start, we take note of the rates in and out of each tank- The total amounts of water in tanks 1 and 2 (30 gallons and 20 gallons, respectively) does not change (important for the rates in and out).

Filling in the quantities from the figure on p. 363, we have:

$$\frac{dQ_1}{dt} = \frac{1.5\text{ gal}}{\text{min}} \cdot \frac{1\text{ oz}}{\text{gal}} + \frac{1.5\text{ gal}}{\text{min}} \cdot \frac{Q_2\text{ oz}}{20\text{ gal}} - \frac{3\text{ gal}}{\text{min}} \cdot \frac{Q_1\text{ oz}}{30\text{ gal}}$$

$$\frac{dQ_2}{dt} = \frac{1\text{ gal}}{\text{min}} \cdot \frac{3\text{ oz}}{\text{gal}} + \frac{3\text{ gal}}{\text{min}} \cdot \frac{Q_1\text{ oz}}{30\text{ gal}} - \frac{4\text{ gal}}{\text{min}} \cdot \frac{Q_2\text{ oz}}{20\text{ gal}}$$

Simplifying, and putting them in order:

$$\frac{dQ_1}{dt} = -\frac{1}{10}Q_1 + \frac{3}{40}Q_2 + \frac{3}{2}$$

$$\frac{dQ_2}{dt} = \frac{1}{10}Q_1 - \frac{1}{5}Q_2 + 3$$

The system is currently non-homogeneous because of the constants 3/2 and 3.

The equilibria are found by solving for where the derivatives are zero. We simplify to make Cramer's Rule easier to apply:

$$\begin{array}{rl} -\frac{1}{10}Q_1 + \frac{3}{40}Q_2 + \frac{3}{2} & = 0 \\ \frac{1}{10}Q_1 - \frac{1}{5}Q_2 + 3 & = 0 \end{array} \quad \Rightarrow \quad \begin{array}{rl} -4Q_1 + 3Q_2 & = -60 \\ Q_1 - 2Q_2 & = -30 \end{array}$$

Therefore, using the determinants, the equilibrium solution is:

$$Q_1 = \frac{210}{5} = 42 \qquad Q_2 = \frac{180}{5}$$

The last part of the question shows that a change of variables results in a homogeneous differential equation. That is, if $x_1 = Q_1 - 42$ and $x_2 = Q_2 - 36$, then substitution into the system of DE's:

$$x_1' = Q_1' \qquad x_2' = Q_2'$$

and

$$-\frac{1}{10}(x_1 + 42) + \frac{3}{40}(x_2 + 36) + \frac{3}{2} = -\frac{1}{10}x_1 + \frac{3}{40}x_2$$

Similarly, substitution into the second equation and substituting:

$$\frac{1}{10}(x_1 + 42) - \frac{1}{5}(x_2 + 36) + 3 = \frac{1}{10}x_1 - \frac{1}{5}x_2$$

Notice that our change of coordinates simply shifted the coordinate system so that the equilibrium is now at the origin. To get the initial conditions, make the last substitution:

$$\begin{array}{rl} x_1' & = -\frac{1}{10}x_1 + \frac{3}{40}x_2 \\ x_2' & = \frac{1}{10}x_1 - \frac{1}{5}x_2 \end{array} \qquad x_1(0) = -17, \quad x_2(0) = -21$$

3. Solve the system of equations given by first converting it into a second order linear ODE (then use Chapter 3 methods):

(a) $\begin{array}{rl} x' & = -2x + y \\ y' & = x - 2y \end{array} \quad \Rightarrow \quad y = x' + 2x \quad \Rightarrow \quad (x' + 2x)' = x - 2(x' + 2x)$

So we have: $x'' + 2x' = x - 2x' - 4x$, or

$$x'' + 4x' + 3x = 0 \quad \Rightarrow \quad r^2 + 4r + 3 = 0 \quad \Rightarrow \quad (r+1)(r+3) = 0$$

Therefore, $x(t) = C_1 e^{-t} + C_2 e^{-3t}$. For y, we have $y = x' + 2x$, or

$$y = -C_1 e^{-t} - 3C_2 e^{-3t} + 2C_1 e^{-t} + 2C_2 e^{-3t} = C_1 e^{-t} - C_2 e^{-3t}$$

(b) $\begin{array}{rl} x' & = 2y \\ y' & = -2x \end{array} \quad \Rightarrow \quad y = \frac{1}{2}x' \quad \Rightarrow \quad (\frac{1}{2}x')' = -2x$

For $x'' + 4x = 0$, we have $r = \pm 2i$, so $x(t) = C_1\cos(2t) + C_2\sin(2t)$, and $y = \frac{1}{2}(-2C_1\sin(2t) + 2C_2\cos(2t)) = -C_1\sin(2t) + C_2\cos(2t)$

4. Give the solution to each system of equations. If it has an infinite number of solutions, give your answer in vector form:

   For the first system,

$$\begin{aligned} 3x + 2y &= 1 \\ 2x - y &= 3 \end{aligned} \quad \Rightarrow \quad x = \frac{\begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix}} = 1 \qquad y = \frac{\begin{vmatrix} 3 & 1 \\ 2 & 3 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix}} = -1$$

   For the second system, the two lines are parallel (NO SOLUTION).

   For the third system, the two lines are the same line, $3x + 2y = 1$, or $y = -(3/2)x + (1/2)$. Writing the equation of the line in vector form, we need a point and a direction. For the point, pick anything- The $y-$intercept, for example, or perhaps the point $(1, -1)$. Secondly, the direction comes from the slope: $-3/2$ is "rise over run", or "change in y over change in x", so the direction can be interpreted as the vector $\langle 2, -3 \rangle$. Therefore, the line (in vector form and parameterized by t):

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} + t \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

5. Let A, B be the matrices below. Compute the matrix operation listed.

$$A = \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

   (a) $2A + B^T = \begin{bmatrix} 4 & -5 \\ 3 & 7 \end{bmatrix}$ (b) $BA = \begin{bmatrix} 0 & -7 \\ 1 & 5 \end{bmatrix}$ (c) $A^{-1} = \frac{1}{7} \begin{bmatrix} 3 & 2 \\ -2 & 1 \end{bmatrix}$

6. Vectors and matrices might have complex numbers. If $z = 3 + 2i$ and vector $\mathbf{v} = [1 + i, 2 - 2i]^T$, then find the real part and the imaginary part of $z\mathbf{v}$.

   SOLUTION:

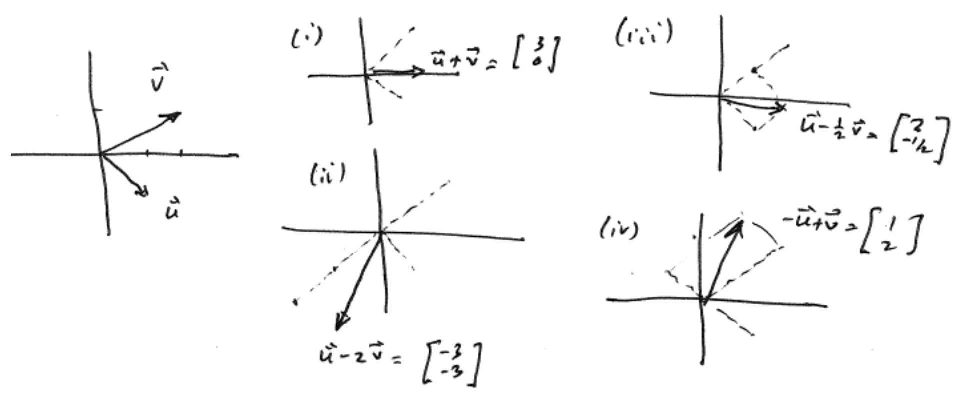
$$z\mathbf{v} = (3 + 2i) \begin{bmatrix} 1 + i \\ 2 - 2i \end{bmatrix} = \begin{bmatrix} 1 + 5i \\ 10 - 2i \end{bmatrix}$$

   The real and imaginary parts of the vector are:

$$\begin{bmatrix} 1 \\ 10 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

7. Adding two vectors: Geometrically (and numerically) compute the following, where $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Be sure to draw each vector out, and see if you can see a pattern.

(a) $\mathbf{u}+\mathbf{v}$ (b) $\mathbf{u}-2\mathbf{v}$ (c) $\mathbf{u}+\frac{1}{2}\mathbf{v}$ (d) $-\mathbf{u}+\mathbf{v}$



8. Verify that $\mathbf{x}_1(t)$ below satisfies the DE: $\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\mathbf{x}$, $\qquad \mathbf{x}_1(t) = \mathrm{e}^{3t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

   SOLUTION: First, the derivative is:

$$\mathbf{x}_1' = 3\mathrm{e}^{3t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

   Secondly, the matrix times $\mathbf{x}_1$ is (I factored the exponential out):

$$\mathrm{e}^{3t}\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \mathrm{e}^{3t}\begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

   And this is the same as the derivative.

9. Consider

$$\begin{aligned} x' &= 2x+3y+1 \\ y' &= x-y-2 \end{aligned} .$$

   First find the equilibrium solution, x_e, y_e. Then show that, if $u = x - x_e$ and $v = y - y_e$, then

$$\begin{aligned} u' &= 2u+3v \\ v' &= u-v \end{aligned}$$

   SOLUTION: This exercise will hopefully make the tank modeling problem make more sense. In this case, solving for the equilibrium solution gives us the system:

$$\begin{aligned} 2x+3y &= -1 \\ x-y &= 2 \end{aligned} \qquad x = 1, y = -1$$

   Now, if $u = x-1$ and $v = y+1$, then

$$\begin{aligned} u' = x' = 2x+3y+1 = 2(u+1)+3(v-1) = 2u+3v \\ v' = y' = x-y-2 = (u+1)-(v-1)-2 = u-v \end{aligned}$$

10. The four graphs in the figure show the direction fields for the four systems of differential equations below. Try to reason out which direction field goes with which system.

(a) $\begin{aligned} x' &= 3x - y \\ y' &= 4x - 2y \end{aligned}$ Graph 3

Along the positive $x-$axis, arrows should go in the direction of $\langle 3, 4\rangle$- Only in Graph 3.

(b) $\begin{aligned} x' &= -2x + y \\ y' &= x - 2y \end{aligned}$ Graph 1

Along the positive $x-$axis, arrows should go in the direction of $\langle -2, 1\rangle$, which could be 1 or 4. Along the positive $y-$axis, arrows should go in the direction of $\langle 1, -2\rangle$, so that is only for Graph 1.

(c) $\begin{aligned} x' &= 2y \\ y' &= -2x \end{aligned}$ Graph 2

Along the positive $x-$axis, arrows are going straight down, so Graph 2 is the only choice.

(d) $\begin{aligned} x' &= -x - 4y \\ y' &= x - y \end{aligned}$ Graph 4

This is 4 by process of elimination, but you can also check the positive $x-$ and $y-$ axis.