# Exercise Set 3 (HW to replace 7.3, 7.5)

This homework is all about solving for eigenvalues and eigenvectors, and we'll also do some visualization and classification of equilibria.

1. Verify that the following function solves the given system of DEs:

$$\mathbf{x}(t) = C_1 \mathrm{e}^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 \mathrm{e}^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \mathbf{x}' = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \mathbf{x}$$

   SOLUTION: It's probably best to break it apart-

$$\begin{array}{l} x_1(t) = C_1\mathrm{e}^{-t} + 2C_2\mathrm{e}^{2t} \\ x_2(t) = 2C_1\mathrm{e}^{-t} + C_2\mathrm{e}^{2t} \end{array} \quad \Rightarrow \quad \begin{array}{l} x_1'(t) = -C_1\mathrm{e}^{-t} + 4C_2\mathrm{e}^{2t} \\ x_2'(t) = -2C_1\mathrm{e}^{-t} + 2C_2\mathrm{e}^{2t} \end{array}$$

   Verify that these expressions for $x_1', x_2'$ are indeed found by

$$x_1' = 3x_1 - 2x_2 \quad \text{and} \quad x_2' = 2x_1 - 2x_2$$

   For example, the first expression below should be $x_1'$ and the second should be $x_2'$

$$\begin{array}{rl} 3x_1(t) & = 3C_1\mathrm{e}^{-t} + 6C_2\mathrm{e}^{2t} \\ -2x_2(t) & = -4C_1\mathrm{e}^{-t} - 2C_2\mathrm{e}^{2t} \\ \hline & = -C_1\mathrm{e}^{-t} + 4C_2\mathrm{e}^{2t} \end{array} \qquad \begin{array}{rl} 2x_1(t) & = 2C_1\mathrm{e}^{-t} + 4C_2\mathrm{e}^{2t} \\ -2x_2(t) & = -4C_1\mathrm{e}^{-t} - 2C_2\mathrm{e}^{2t} \\ \hline & = -2C_1\mathrm{e}^{-t} + 2C_2\mathrm{e}^{2t} \end{array}$$

   And those do check out.

2. For each matrix, find the eigenvalues and eigenvectors. Note that they may be complex (when solving the quadratic equation).

   (a) $A = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} \quad \Rightarrow \quad \lambda_1 = 4, \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \lambda_2 = 2, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

   (b) $A = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$
   SOLUTION: $\lambda^2 - 2\lambda + 5 = 0$, so $\lambda^2 - 2\lambda + 1 = -4$ and $\lambda = 1 \pm 2i$.
   For $\lambda = 1 + 2i$, we have:

$$(3 - (1 + 2i))v_1 - 2v_2 = 0 \quad \Rightarrow \quad (2 - 2i)v_1 - 2v_2 = 0 \quad \Rightarrow \quad (1 - i)v_1 - v_2 = 0$$

   so we can take $\mathbf{v} = \begin{bmatrix} 1 \\ 1 - i \end{bmatrix}$. For $\lambda = 1 - 2i$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 + i \end{bmatrix}$, which is the complex conjugate (that always happens for real matrices $A$).

   (c) $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \quad \Rightarrow \quad \lambda_1 = -3, \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \lambda_2 = -1, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

   (d) $A = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix} \quad \Rightarrow \quad \lambda_1 = 2 + i, \mathbf{v}_1 = \begin{bmatrix} -2 \\ 1 + i \end{bmatrix} \quad \text{and} \quad \lambda_2 = 2 - i, \mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 - i \end{bmatrix}$

   (e) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
   For this one, we see that $\lambda = 1, 1$, but when we solve for the eigenvector, we just get $0 = 0$. That means that the values $v_1, v_2$ can each be any number (two free variables). Therefore, $\mathbf{v}_1, \mathbf{v}_2$ can be any vectors in the plane, as long as neither is zero and they are not multiples of each other.

(f) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Similar to the last problem, $\lambda = 1, 1$, but there is only one eigenvector $v_2 = 0$, so $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

3. Convert each of the systems $\mathbf{x}' = A\mathbf{x}$ into a single second order differential equation, and solve it using methods from Chapter 3, if $A$ is given below:

(a) $A = \begin{bmatrix} 1 & 2 \\ -5 & -1 \end{bmatrix}$

SOLUTION: Substitute $x_2 = \frac{1}{2}(x_1' - x_1)$ into the second equation to get $x_1'' + 9x_1 = 0$, so

$$x_1(t) = C_1 \cos(3t) + c_2 \sin(3t)$$

Then put it back into the expression above to determine $x_2$:

$$x_2(t) = \frac{3C_2 - C_1}{2} \cos(3t) - \frac{3C_1 + C_2}{2} \sin(3t)$$

(b) $A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$

SOLUTION: Substitute $x_2 = x_1' - x_1$ into the second equation, and find that $x_1'' - 2x_1' - 3x_1 = 0$. From that,

$$x_1(t) = C_1 \mathrm{e}^{-t} + C_2 \mathrm{e}^{3t} \qquad x_2(t) = -2C_1 \mathrm{e}^{-t} + 2C_2 \mathrm{e}^{3t}$$

(c) $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$

SOLUTION: Note that its easier to start with the second equation rather than the first: $x_1 = -(x_2' + x_2)$. This gives us an IVP in $x_2$:

$$x_2'' - 2x_2' + x_2 = 0 \quad \Rightarrow \quad x_2(t) = \mathrm{e}^t (C_1 + C_2 t)$$

Then

$$x_1 = -2\mathrm{e}^t (C_1 + C_2 t) - C_2 \mathrm{e}^t$$

4. Consider the expression: $\mathrm{e}^{\lambda t}\mathbf{v}$, where $\mathbf{v}$ is a two dimensional non-zero vector that we'll assume is fixed. Below we want to consider what happens graphically as we change $\lambda$.

- If $\lambda < 0$, what happens specifically as $t \to \infty$? What happens as $t \to -\infty$?
- If $\lambda = 0$, what happens specifically as $t \to \infty$? What happens as $t \to -\infty$?
- If $\lambda > 0$, what happens specifically as $t \to \infty$? What happens as $t \to -\infty$?

SOLUTION: The expression $\mathrm{e}^{\lambda t}\mathbf{v}$, with $\lambda$ fixed and $t$ varying, is a "ray" extending outward from the origin. If $\lambda < 0$, then $\mathrm{e}^{\lambda t}\mathbf{v} \to (0, 0)$ along the ray. similarly, if $\lambda > 0$, $\mathrm{e}^{\lambda t}\mathbf{v} \to \infty$ along the ray. If $\lambda = 0$, there is no $t$ and the solution is fixed at $\mathbf{v}$.

5. Give the general solution to each system $\mathbf{x}' = A\mathbf{x}$ using eigenvalues and eigenvectors, and sketch a phase plane (solutions in the $x_1, x_2$ plane). Identify the origin as a *sink*, *source* or *saddle*:

*NOTE: We didn't go through the sketching of solutions until Wed, Dec 4th, but after that you should be able to answer the questions.*

(a) $A = \begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}$ The trace is 2, determinant is -24, so $\lambda^2 - 2\lambda - 24 = 0$ and $\lambda = -4, 6$.

For $\lambda = -4$, we have $(1 - -4)v_1 + 5v_2 = 0$, or $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

For $\lambda = 6$, we have $(1 - 6)v_1 + 5v_2 = 0$, or $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

The general solution is:

$$\mathbf{x}(t) = C_1 e^{-4t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + C_2 e^{6t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(b) $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$ We should find $\lambda = 3, 5$.

For $\lambda = 3$, we have $(7 - 3)v_1 + 2v_2 = 0$, or $\mathbf{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

For $\lambda = 5$, we have $(7 - 5)v_1 + 2v_2 = 0$, or $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

$$\mathbf{x}(t) = C_1 e^{3t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + C_2 e^{5t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

DISCUSSION: Since both eigenvalues are positive, the origin is a SOURCE, but to graph it, think about time running in reverse. Then solutions will go to the origin tangent to $\langle 1, -2 \rangle$ (with reversed arrows).

(c) $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$ We should find $\lambda = -2, 5$.

For $\lambda = -2$, we have $(2 - -2)v_1 + 3v_2 = 0$, or $\mathbf{v} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$.

For $\lambda = 5$, we have $(2 - 5)v_1 + 3v_2 = 0$, or $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$$\mathbf{x}(t) = C_1 e^{-2t} \begin{bmatrix} 3 \\ -4 \end{bmatrix} + C_2 e^{5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(d) $A = \begin{bmatrix} -1 & 0 \\ 3 & -2 \end{bmatrix}$ We should find $\lambda = -1, -2$.

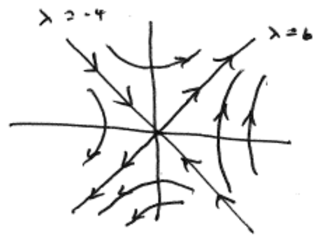
For $\lambda = -1$, we have $(-1 + 1)v_1 + 0v_2 = 0$, so we just get $0 = 0$- Use the other equation to get $3v_1 + (-2 + 1)v_2 = 0$, or $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

For $\lambda = -2$, we have $(-1 + 2)v_1 + 0v_2 = 0$, or $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (negative not necessary, but you can use it if you like)

$$\mathbf{x}(t) = C_1 e^{-t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + C_2 e^{-2t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

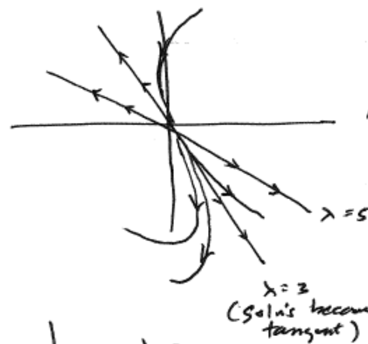
For sketches, see the next page!

$$\vec{X} = c_1 e^{-4t}\begin{bmatrix}1\\-1\end{bmatrix} + c_2 e^{6t}\begin{bmatrix}1\\1\end{bmatrix}$$



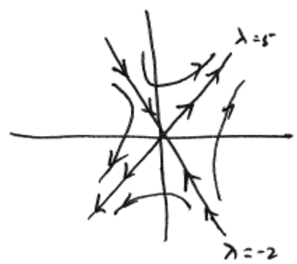

"Saddle"

$$\vec{X} = c_1 e^{3t}\begin{bmatrix}1\\-2\end{bmatrix} + c_2 e^{5t}\begin{bmatrix}1\\-1\end{bmatrix}$$



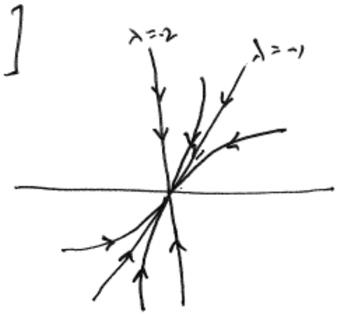

"Source"

$$\vec{X} = c_1 e^{-2t}\begin{bmatrix}3\\-4\end{bmatrix} + c_2 e^{5t}\begin{bmatrix}1\\1\end{bmatrix}$$




"saddle"

$$\vec{X} = \underbrace{c_1 e^{-t}\begin{bmatrix}1\\3\end{bmatrix}}_{\text{Tangent "line"}} + c_2 e^{-2t}\begin{bmatrix}0\\1\end{bmatrix}$$




"Sink"