Summary- Elements of Chapters 7 and 9

We started by looking at general systems of equations. Be able to convert an n^{th} order DE to a system of first order. Be able to convert a system of two first order equations to a single equation of second order. Be able to convert a system to dy/dx form. (See "Homework, Day 1 (Conversions)" on the class website).

Eigenvalues and Eigenvectors

For the following, we are solving the system:

$$\begin{aligned} x' &= ax + by \\ y' &= cx + dy \end{aligned} \quad \Leftrightarrow \quad \begin{bmatrix} x' \\ y' \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad \Leftrightarrow \quad \mathbf{x}' = A\mathbf{x}$$

1. Definition: If there is a constant λ and a non-zero vector $\mathbf{v}$ that solves

$$\begin{array}{rrl} (a-\lambda)v_1 & +bv_2 & = 0 \\ cv_1 & +(d-\lambda)v_2 & = 0 \end{array} \tag{1}$$

then λ is an eigenvalue, and $\mathbf{v}$ is an associated eigenvector. This system has a non-zero solution for v_1, v_2 only if the two lines are multiples of each other. In that case, the determinant must be zero.

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad \lambda^2 - (a+d)\lambda + (ad - bc) = 0 \quad \Rightarrow \lambda^2 - \text{Tr}(A)\lambda + \det(A) = 0$$

And this is the **characteristic equation**. This is formallly solved via the quadratic formula, but we would typically factor it or complete the square. For each λ, we must go back and solve Equation (**??**) to find $\mathbf{v}$. As a shortcut, the eigenvector can be written down directly (as long as the equation is not $0 = 0$)

$$(a-\lambda)v_1 + cv_2 = 0 \quad \Rightarrow \quad \mathbf{v} = \begin{bmatrix} -c \\ a-\lambda \end{bmatrix}$$

Solve $\mathbf{x}' = A\mathbf{x}$

1. We make the ansatz: $\mathbf{x}(t) = \mathrm{e}^{\lambda t}\mathbf{v}$, substitute into the DE, and we find that $\lambda, \mathbf{v}$ must be an eigenvalue, eigenvector of the matrix A.

2. The eigenvalues are found by solving the characteristic equation:

$$\lambda^2 - \text{Tr}(A)\lambda + \det(A) = 0 \qquad \lambda = \frac{\text{Tr}(A) \pm \sqrt{\Delta}}{2}$$

The solution is one of three cases, depending on Δ:

- Real λ_1, λ_2 with two eigenvectors, $\mathbf{v}_1, \mathbf{v}_2$:

$$\mathbf{x}(t) = C_1\mathrm{e}^{\lambda_1 t}\mathbf{v}_1 + C_2\mathrm{e}^{\lambda_2 t}\mathbf{v}_2$$

- Complex $\lambda = a + ib$, $\mathbf{v}$ (we only need one):

$$\mathbf{x}(t) = C_1\text{Re}\left(\mathrm{e}^{\lambda t}\mathbf{v}\right) + C_2\text{Im}\left(\mathrm{e}^{\lambda t}\mathbf{v}\right)$$

- One eigenvalue, one eigenvector (which is not needed). Determine $\mathbf{w}$, where:

$$\begin{aligned} (a-\lambda)x_0 + cy_0 &= w_1 \\ cx_0 + (d-\lambda)y_0 &= w_2 \end{aligned}$$

Then

$$\mathbf{x}(t) = \mathrm{e}^{\lambda t}\left(\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + t\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}\right) = \mathrm{e}^{\lambda t}(\mathbf{x}_0 + t\mathbf{w})$$

Note: In this solution, we don't have arbitrary constants- rather, we're writing the solution in terms of the initial conditions.

You might find this helpful- Below there is a chart comparing the solutions from Chapter 3 to the solutions in Chapter 7:

	Chapter 3	Chapter 7
Form:	$ay'' + by' + cy = 0$	$\mathbf{x}' = A\mathbf{x}$
Ansatz:	$y = \mathrm{e}^{rt}$	$\mathbf{x} = \mathrm{e}^{\lambda t}\mathbf{v}$
Char Eqn:	$ar^2 + br + c = 0$	$\det(A - \lambda I) = 0$
Real Solns	$y = C_1\mathrm{e}^{r_1 t} + C_2\mathrm{e}^{r_2 t}$	$\mathbf{x}(t) = C_1\mathrm{e}^{\lambda_1 t}\mathbf{v}_1 + C_2\mathrm{e}^{\lambda_2 t}\mathbf{v}_2$
Complex	$y = C_1\mathrm{Re}(\mathrm{e}^{rt}) + C_2\mathrm{Im}(\mathrm{e}^{rt})$	$\mathbf{x}(t) = C_1\mathrm{Re}\left(\mathrm{e}^{\lambda t}\mathbf{v}\right) + C_2\mathrm{Im}\left(\mathrm{e}^{\lambda t}\mathbf{v}\right)$
SingleRoot	$y = \mathrm{e}^{rt}(C_1 + C_2 t)$	$\mathbf{x}(t) = \mathrm{e}^{\lambda t}\left(\mathbf{x}_0 + t\mathbf{w}\right)$

Classification of the Equilibria

The origin is always an equilibrium solution to $\mathbf{x}' = A\mathbf{x}$, and we can use the Poincaré Diagram to help us classify the origin (in Chapter 7) or other equilibrium solutions (in Chapter 9).

Solve General Nonlinear Equations

We don't have a method that will work on every system of nonlinear differential equations, although there are some tricks we can try with special cases- that is, given the system

$$\begin{aligned} \tfrac{dx}{dt} &= f(x,y) \\ \tfrac{dy}{dt} &= g(x,y) \end{aligned} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{g(x,y)}{f(x,y)}$$

And we might get lucky if it is in the form of an equation from Chapter 2.

Local Analysis of Nonlinear Equations

Often, we can perform a local analysis of a system of nonlinear DEs by "linearizing about the equilibria". Given

$$\begin{aligned} \tfrac{dx}{dt} &= f(x,y) \\ \tfrac{dy}{dt} &= g(x,y) \end{aligned}$$

- Find the equilibrium solutions ($f(x,y) = 0$ and $g(x,y) = 0$).
- At each equilibrium, we perform the local analysis by first linearizing, then we classify the equilibrium. Given an equilibrium at $x = a, y = b$, we construct the matrix (the Jacobian) at that point:

$$\begin{bmatrix} f_x(a,b) & f_y(a,b) \\ g_x(a,b) & g_y(a,b) \end{bmatrix}$$

  Use the Poincaré Diagram to classify the equilibrium.

Modeling

Recall that we also did some modeling in these sections- Primarily, we looked at the predator-prey model and the tank mixing problem (with multiple tanks). Given a system that represents two populations, you should be able to determine if the system represents predator-prey, competing species, or cooperating species.