Class Notes from Nov 13

Today we'll solve

$$y'' - xy' - y = 0 \qquad y(1) = 1 \quad y'(1) = 2$$

using the power series solution for $y(x)$:

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n = \sum_{n=0}^{\infty} a_n (x-1)^n$$

Notice that "to solve" the DE means to compute the coefficients a_n.

Solve by Differentiation

In the first of two methods, we compute the coefficients a_n using the Taylor series formula:

$$a_n = \frac{y^{(n)}(x_0)}{n!}$$

We showed that $a_0 = y(x_0)$ and $a_1 = y'(x_0)$, so those are always computed by the IVP. In our case,

$$a_0 = y(1) = 1 \qquad a_1 = y'(1) = 2$$

To compute the others, notice that

$$y'' = xy' + y \quad \Rightarrow \quad y''(1) = y'(1) + y(1) = 2 + 1 = 3 \quad \Rightarrow \quad a_2 = \frac{3}{2!}$$

Now differentiate both sides to compute $y'''(1)$. Remember the product rule!

$$y''' = y' + xy'' + y' = 2y' + xy'' \quad \Rightarrow \quad y'''(1) = 2y'(1) + y''(1) = 7 \quad \Rightarrow \quad a_3 = \frac{7}{3!}$$

Now we compute a_4:

$$y^{(4)} = 3y'' + xy''' \quad \Rightarrow \quad y^{(4)}(1) = 3y''(1) + y'''(1) = 16 \quad \Rightarrow \quad a_4 = \frac{16}{4!}$$

If we're asked to find the first five terms (which is typical), we'll stop here and write:

$$y(x) = 1 + 2(x-1) + \frac{3}{2!}(x-1)^2 + \frac{7}{3!}(x-1)^3 + \frac{16}{4!}(x-1)^4 + \cdots$$

Solve by Substitution

Using this technique, we solve for all of the coefficients recursively (we'll see what that means later). For now, we start with our ansatz and substitute it into the differential equation:

$$y = \sum_{n=0}^{\infty} a_n (x-1)^n \qquad y' = \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} \qquad y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2}$$

Now, the differential equation $y'' - xy' - y = 0$ becomes:

$$\sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} - x\sum_{n=1}^{\infty} na_n(x-1)^{n-1} - \sum_{n=0}^{\infty} a_n(x-1)^n = 0$$

Our goal is to write this as a SINGLE sum:

$$\sum \left(\qquad\qquad C_m \qquad\qquad \right)(x-1)^m = 0,$$

because at that point, we can conclude that every coefficient C_m must be zero: $C_m = 0$ for all m. We'll see where that takes us, but first let's get that single sum.

Start by getting all the variables inside the relevant sums. In our case, we need to work on the middle sum first. We bring the x into the sum by rewriting it first:

$$x\sum_{n=1}^{\infty} na_n(x-1)^{n-1} = [(x-1)+1]\sum_{n=1}^{\infty} na_n(x-1)^{n-1} =$$

$$(x-1)\sum_{n=1}^{\infty} na_n(x-1)^{n-1} + \sum_{n=1}^{\infty} na_n(x-1)^{n-1} = \sum_{n=1}^{\infty} na_n(x-1)^n + \sum_{n=1}^{\infty} na_n(x-1)^{n-1}$$

We were able to "simplify" by making the one sum into two sums. Now, we want to bring all the sums together- To do that, **make sure that the powers all begin with the same index**. We'll go sum by sum below and show what substitutions to make:

$$\sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} \quad \text{Starts with } (x-1)^0 \qquad \text{Let } m = n-2$$

$$\sum_{n=1}^{\infty} na_n(x-1)^n \quad \text{Starts with } (x-1)^1 \quad \Rightarrow \quad \sum_{n=0}^{\infty} na_n(x-1)^n$$

(notice that I just dropped the index to start at 0 which did not add anything to the sum (except that we added zero). For this sum, we can let $m = n$. For the other middle sum:

$$\sum_{n=1}^{\infty} na_n(x-1)^{n-1} \quad \text{Starts with } (x-1)^0 \qquad \text{Let } m = n-1$$

And the last sum was just the original sum:

$$\sum_{n=0}^{\infty} a_n(x-1)^n \quad \text{Starts with}(x-1)^0 \quad \Rightarrow \quad \text{Let } m = n$$

Now the differential equation becomes:

$$y'' - xy' - y = 0 \quad \Rightarrow \quad \sum_{m=0}^{\infty} \left((m+2)(m+1)a_{m+2} - ma_m - (m+1)a_{m+1} - a_m\right)(x-a)^m = 0$$

This polynomial (or series) is zero for all x, so therefore the coefficients must be zero:

$$(m+2)(m+1)a_{m+2} - (m+1)a_{m+1} - (m+1)a_m = 0 \quad \Rightarrow \quad a_{m+2} = \frac{a_{m+1} + a_m}{m+2}$$

This is called the **recurrence relation**. We can compute the coefficients from here, using $a_0 = 1$ and $a_1 = 2$. Let's check that we get the same coefficients as before.

$$a_2 = \frac{a_1 + a_0}{2} = \frac{3}{2}$$

$$a_3 = \frac{a_2 + a_1}{3} = \left(\frac{3}{2} + 2\right) \cdot \frac{1}{3} = \frac{7}{2} \cdot \frac{1}{3} = \frac{7}{3!}$$

$$a_4 = \frac{a_3 + a_2}{4} = \left(\frac{7}{6} + \frac{3}{2}\right) \cdot \frac{1}{4} = \frac{16}{6} \cdot \frac{1}{4} = \frac{16}{4!}$$

And those are the same values as before. This is as far as we'll go on this exam...

Quiz 8

1. Find the radius and interval of convergence for the series $\displaystyle\sum_{n=1}^{\infty} \frac{(x-1)^n}{n3^n}$
2. Exercise #24 in 5.1.
3. For the IVP in #8, Section 5.2, find the recurrence relation (as we did above).
4. For $y'' + \sin(x)y' + \cos(x)y = 0$, find the first five terms of the power series expansion for $y(x)$ as we did in the first section of these notes. (Note: Zeros do count as a term)

Please write your solutions NEATLY and turn them in on Friday. We're doing the quiz this way to give you some extra motivation to do the homework before the weekend!