Math 244 Extra Practice: Conversions

We stopped our consideration of second order DEs- We'll re-visit it later, but we want to pause for a moment to show that all n^{th} order differential equations can be converted to a system of first order DEs. We can also go backwards for some systems- Converting a system to an n^{th} order DE.

We'll show you how to do the conversions via some examples.

1. Convert $y'' = ty' + 3y$ into a system of first order.

   SOLUTION: We introduce new variables- Let $x_1 = y$ and $x_2 = y'$. We write a system using x_1 and x_2 (note that x_1, x_2 are functions of t)

   $$\begin{array}{ll} x_1' & = y' = x_2 \\ x_2' & = y'' = ty' + 3y = 3x_1 + tx_2 \end{array} \quad \Rightarrow \quad \begin{array}{ll} x_1' & = x_2 \\ x_2' & = 3x_1 + tx_2 \end{array}$$

   (Notice that the answer should be all in terms of the new variables)

2. Convert $2y'' + 4y' + 3y = 0$ into a system of first order DEs:

   SOLUTION: We introduce new variables- Let $x_1 = y$ and $x_2 = y'$. We write a system using x_1 and x_2 (note that x_1, x_2 are functions of t)

   $$\begin{array}{ll} x_1' & = y' = x_2 \\ x_2' & = y'' = -\frac{3}{2}y - 2y' = -\frac{3}{2}x_1 - 2x_2 \end{array} \quad \Rightarrow \quad \begin{array}{ll} x_1' & = x_2 \\ x_2' & = -\frac{3}{2}x_1 - 2x_2 \end{array}$$

3. Convert $y''' = 3y'' - 2y' + y + t^2$ into a system of first order DEs.

   SOLUTION: We'll need three variables now: Let $x_1 = y$, $x_2 = y'$ and $x_3 = y''$. Then forming the system of differential equations proceeds much as before:

   $$\begin{array}{ll} x_1' & = y' = x_2 \\ x_2' & = y'' = x_3 \\ x_3' & = y''' = 3y'' - 2y' + y + t^2 = 3x_3 - 2x_2 + x_1 + t^2 \end{array} \quad \Rightarrow \quad \begin{array}{ll} x_1' & = x_2 \\ x_2' & = x_3 \\ x_3' & = 3x_3 - 2x_2 + x_1 + t^2 \end{array}$$

4. Convert $y^{(iv)} = y + y' + y'' + y'''$ to a system of first order DEs.

   SOLUTION: We need 4 variables: x_1, x_2, x_3, x_4, and we end up with:

   $$\begin{array}{ll} x_1' & = x_2 \\ x_2' & = x_3 \\ x_3' & = x_4 \\ x_4' & = x_1 + x_2 + x_3 + x_4 \end{array}$$

Going Backwards: A system to second order

Given a linear system with constant coefficients, convert to an equivalent second order linear homogeneous DE.

The idea will be that the first equation will be used to substitute into the second so that we end up with an equation all in x_2 (or equivalently, all in x_1).

$$\begin{aligned} x_1' &= ax_1 + bx_2 \\ x_2' &= cx_1 + dx_2 \end{aligned} \quad \Rightarrow \quad x_2 = \frac{1}{b}x_1' - \frac{a}{b}x_1$$

Now use the second equation and subsititute:

$$x_2' = cx_1 + dx_2 \quad \Rightarrow \quad (\frac{1}{b}x_1' - \frac{a}{b}x_1)' = cx_1 + (\frac{1}{b}x_1' - \frac{a}{b}x_1)$$

Notice that this is all in terms of x_1. Here are a couple of numerical examples:

1. Convert to an equivalent 2d order DE: $\begin{aligned} x_1' &= 3x_1 + x_2 \\ x_2' &= -x_1 + 2x_2 \end{aligned}$

   SOLUTION: Using the first equation, we have $x_2 = x_1' - 3x_1$. Substitute into the second to get:

   $$(x_1' - 3x_1)' = -x_1 + 2(x_1' - 3x_1) \quad \Rightarrow \quad x_1'' - 3x_1' = -x_1 + 2x_1' - 6x_1$$

   Simplifying, we get

   $$x_1'' - 5x_1' + 7x_1 = 0$$

   And this is our second order DE. As a "side remark", if I had used the second equation to substitute into the first equation, I would have ended up with the same second order DE (in x_2 instead of x_1).

2. Convert to an equivalent 2d order DE: $\begin{aligned} x_1' &= x_1 + \tfrac{1}{2}x_2 \\ x_2' &= 2x_1 - 3x_2 \end{aligned}$

   SOLUTION: Using the first equation, we have $x_2 = 2x_1' - 2x_1$. Substitute into the second to get:

   $$(2x_1' - 2x_1)' = 2x_1 - 3(2x_1' - 2x_1) \quad \Rightarrow \quad 2x_1'' - 2x_1' = 2x_1 - 6x_1' + 6x_1$$

   Simplifying, we get

   $$2x_1'' + 4x_1' - 8x_1 = 0$$

   And this is our second order DE.

Exercises

1. Convert the following n^{th} order DEs to systems of first order.

   (a) $y'' + 3y' + 2y = 0$

   (b) $y''' = 2y - 3y'$

   (c) $2y'' + 6y' + y = 0$

   (d) $y'' + 5y = 0$

   (e) $y^{(v)} = y - 3y' + ty'' + y''' - 3y^{(iv)}$

2. Convert the following systems of first order to an equivalent second order DE, if possible.

   (a) $\begin{aligned} x_1' &= x_1 + 2x_2 \\ x_2' &= 2x_1 + x_2 \end{aligned}$

   (b) $\begin{aligned} x_1' &= -2x_1 + x_2 \\ x_2' &= x_1 + x_2 \end{aligned}$

   (c) $\begin{aligned} x_1' &= x_1 \\ x_2' &= 3x_2 \end{aligned}$

3. Convert the system to a single first order equation by expressing it as follows, then solve the differential equation.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

   (a) $\begin{aligned} x' &= y \\ y' &= -x \end{aligned}$

   (b) $\begin{aligned} x' &= x \\ y' &= (x - 3y)x \end{aligned}$

   (c) $\begin{aligned} x' &= x - 3y^2 \\ y' &= -(2x + y) \end{aligned}$

4. Solve the de-coupled system in 2(c).