Solutions to the Extra Practice in conversions:

1. Convert the following n^{th} order DEs to systems of first order.

   (a) $y'' + 3y' + 2y = 0$
   SOLUTION: Let $x_1 = y, x_2 = y'$. Then:

   $$\begin{aligned} x_1' &= x_2 \\ x_2' &= -2x_1 - 3x_2 \end{aligned}$$

   (b) $y''' = 2y - 3y'$
   SOLUTION: Let $x_1 = y, x_2 = y', x_3 = y''$. Then:

   $$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= 2x_1 - 3x_2 \end{aligned}$$

   (c) $2y'' + 6y' + y = 0$
   SOLUTION: Let $x_1 = y, x_2 = y'$. Then:

   $$\begin{aligned} x_1' &= x_2 \\ x_2' &= -(1/2)x_1 - 3x_2 \end{aligned}$$

   (d) $y'' + 5y = 0$
   SOLUTION: Let $x_1 = y, x_2 = y'$. Then:

   $$\begin{aligned} x_1' &= x_2 \\ x_2' &= -5x_1 \end{aligned}$$

   (e) $y^{(v)} = y - 3y' + ty'' + y''' - 3y^{(iv)}$
   SOLUTION: Let $x_1 = y, x_2 = y', x_3 = y'', x_4 = y''', x_5 = y^{(iv)}$. Then:

   $$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= x_4 \\ x_4' &= x_5 \\ x_5' &= x_1 - 3x_2 + tx_3 + x_4 - 3x_5 \end{aligned}$$

2. Convert the following systems of first order to an equivalent second order DE, if possible.

   (a) $\begin{aligned} x_1' &= x_1 + 2x_2 \\ x_2' &= 2x_1 + x_2 \end{aligned}$ Using Eqn 1, get $x_2 = \frac{1}{2}x_1' - \frac{1}{2}x_1$:

   $$\frac{1}{2}x_1'' - \frac{1}{2}x_1' = 2x_1 + \frac{1}{2}x_1' - \frac{1}{2}x_1 \quad \Rightarrow \quad x_1'' - 2x_1' - 3x_1 = 0$$

(b) $\begin{aligned} x_1' &= -2x_1 + x_2 \\ x_2' &= x_1 + x_2 \end{aligned}$ Using Eqn 1, get $x_2 = x_1' + 2x_1$:

$$x_1'' + 2x_1' = x_1 + x_1' + 2x_1 \quad \Rightarrow \quad x_1'' + x_1' - 3x_1 = 0$$

(c) $\begin{aligned} x_1' &= x_1 \\ x_2' &= 3x_2 \end{aligned}$

Since the DE for x_1 depends only on x_1 and the DE for x_2 depends only on x_2, we say that the system is **decoupled**. In that case, we cannot express the system as a single second order DE.

3. Convert the system to a single first order equation by expressing it as follows, then solve the differential equation.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

(a) $\begin{aligned} x' &= y \\ y' &= -x \end{aligned}$

SOLUTION:

$$\frac{dy}{dx} = \frac{-x}{y} \quad \Rightarrow \quad \int y\,dy = -\int x\,dx \quad \Rightarrow \quad \frac{1}{y}y^2 = -\frac{1}{2}x^2 + C \quad \Rightarrow \quad x^2 + y^2 = k$$

Solutions in the $xy-$plane are circles.

(b) $\begin{aligned} x' &= x \\ y' &= (x - 3y)x \end{aligned}$

SOLUTION:

$$\frac{dy}{dx} = \frac{(x-3y)x}{x} \quad \Rightarrow \quad \frac{dy}{dx} = x - 3y \quad \Rightarrow \quad y' + 3y = x$$

This is a linear differential equation with integrating factor e^{3x}.

$$(ye^{3x})' = xe^{3x}$$

Use integration by parts to get:

$$(ye^{3x} = \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + C \quad \Rightarrow \quad y = \frac{1}{3}x - \frac{1}{9} + Ce^{-3x}$$

NOTE: You also could have solved this using the homogeneous and particular solutions with Method of Undetermined Coefficients.

(c) $\begin{aligned} x' &= x - 3y^2 \\ y' &= -(2x + y) \end{aligned}$

SOLUTION: Looking at the terms, we can't separate variables and it is not linear, so it may be exact:

$$\frac{dy}{dx} = \frac{-(2x+y)}{x - 3y^2} \quad \Rightarrow \quad (2x + y) + (x - 3y^2)\frac{dy}{dx} = 0$$

Now, $N_y = 1 = M_x$. We should get that the solution is:

$$x^2 + xy - y^3 = C$$

4. Solve the de-coupled system in 2(c).

   SOLUTION: The two differential equations can be solved independently:

$$x_1' = x_1 \quad \Rightarrow x_1 = C_1 \mathrm{e}^t \quad \text{and} \quad x_2' = 3x_2 \quad \Rightarrow x_2 = C_2 \mathrm{e}^{3t}$$

   As a set of parametric functions,

$$\mathbf{x}(t) = \langle C_1 \mathrm{e}^t, C_2 \mathrm{e}^{3t} \rangle$$