Last time:

- Vocab: ODE, PDE, IVP, Order of a DE, Solution to a DE
- Skills: Be able to verify that $\phi(t)$ is a solution to a DE.
- Three models: Mice/Owls, Newton's Law of Cooling.
- if $dy/dt = ky$, then $y(t) = Ae^{kt}$

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$$\frac{d}{dx}(\sinh(x)) = \cosh(x) \qquad \frac{d}{dx}(\cosh(x)) = \sinh(x)$$

Review

Let's review "integration by parts"! Usual way (inverse product rule)

$$\int u\,dv = uv - \int v\,du$$

However, if we have to perform integration by parts several times, a table is more useful (handout and boardwork).

Today in Differential Equations:

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- Be able to visualize solutions to general DE: $y' = f(t, y)$.

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or

$$y + \frac{b}{a} = C\mathrm{e}^{at} \quad \Rightarrow \quad y = C\mathrm{e}^{at} - \frac{b}{a}$$

where C depends on the initial conditions...

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What do the solutions do for large t?
We note that for any C, the solution will converge to $5/2$ as t gets large.

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The DE has an **equilibrium** solution if there is a constant solution to the DE.
We can find it by setting the derivative to zero...
For $y' = ay + b$, $y(t) = -b/a$ is the equilibrium solution.

Visualizing Solutions - General Case

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$$y' = f(t, y)$$

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Definition: A **direction field** is a plot in the (t, y) plane that give the local tangent lines to the solution to a first order ODE.

Example: $y' = t - y^2$

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t	y	$t - y^2$
1	-1	

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t	y	$t - y^2$
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2	1	1
-1	1	-2

Isoclines:

In drawing a picture, we might consider curves of constant slope. For example, with zero slope:

$$0 = t - y^2 \quad \Rightarrow \quad y^2 = t$$

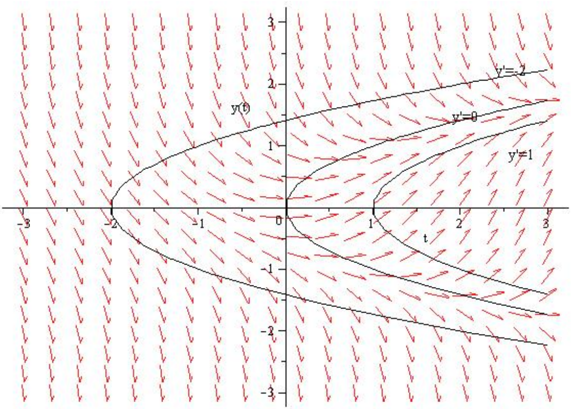
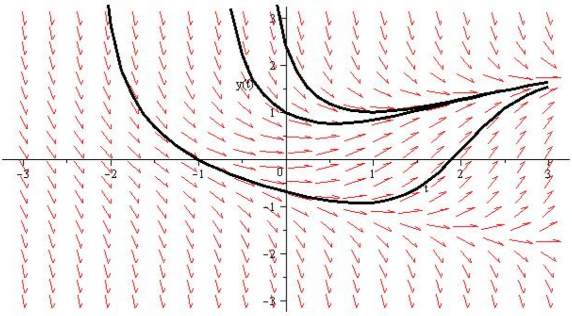
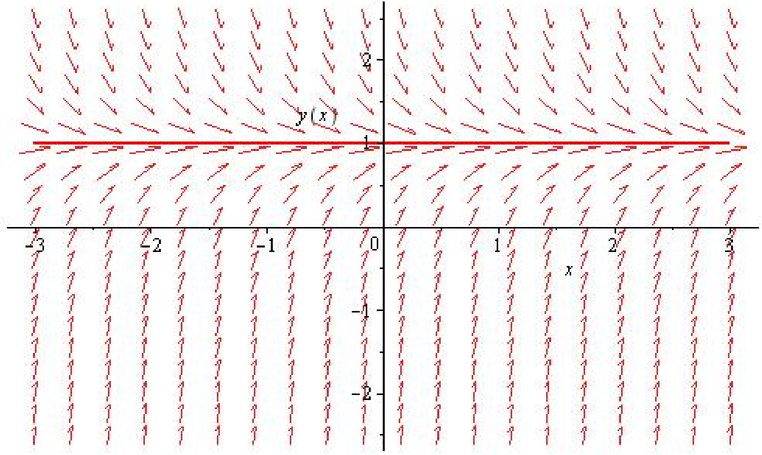


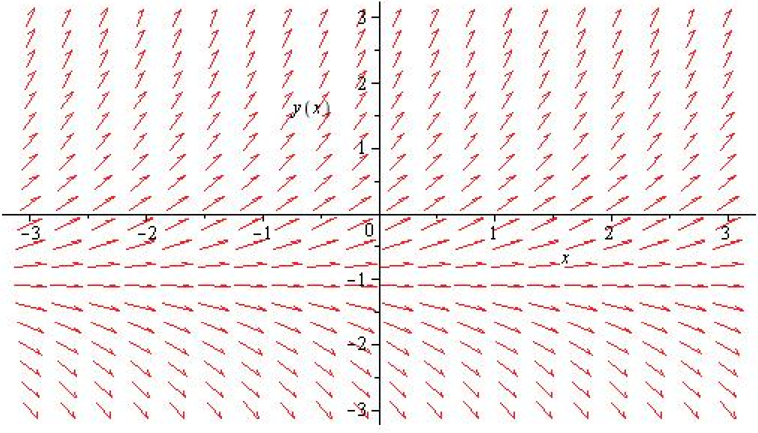
Figure: Direction Field with Isoclines: $y' = -2, y' = 0, y' = 1$


y(t)
t
3
2
1
-1
-2
-3
-3
-2
-1
0
1
2
3

Give an ODE of the form $y' = ay + b$ whose direction field looks like:

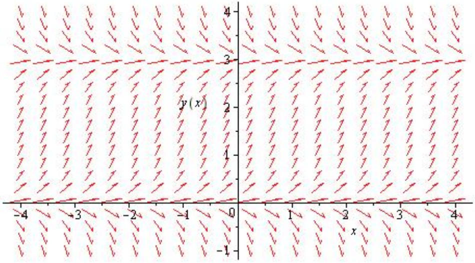


Same question as before:



Choose a DE

1. $y' = 3 - y$
2. $y' = y(y + 3)$
3. $y' = y(3 - y)$
4. $y' = 2y - 1$



Homework Hint: #22, Section 1.1

$$V = \frac{4}{3}\pi r^3 \qquad A = 4\pi r^2$$

so if $V' = kA$, give V' in terms of V only.

Homework Hint: #14, Section 1.3

Differentiate the following with respect to t:

$$f(t)\int_0^t G(s)\,ds$$

SOLUTION: Use the product rule and the FTC:

$$f'(t)\int_0^t G(s)\,ds + f(t)G(t)$$

Section 2.1: Linear DEs

Definition: Linear first order ODE is any DE that can be expressed as:

$$\frac{dy}{dt} + a(t)y(t) = f(t) \quad \text{or} \quad y' + a(t)y = f(t)$$

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Question: Is there a function $e^{P(t)}$ that will turn the left side of the DE to the derivative of something?

Solve Linear DEs using Integrating Factor

Given $y' + a(t)y = f(t)$, we compute the **integrating factor**

$$e^{\int a(t)\,dt}$$

and multiply the DE by it:

$$e^{\int a(t)\,dt}(y' + a(t)y) = f(t)e^{\int a(t)\,dt}$$

This makes the left side a single derivative:

$$\left(y(t)e^{\int a(t)\,dt}\right)' = f(t)e^{\int a(t)\,dt}$$

which can be solved by integrating both sides.

$$y(t)e^{\int a(t)\,dt} = \int f(t)e^{\int a(t)\,dt}\,dt$$

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(Remember to include the constant of integration!)

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(Remember to include the constant of integration!)
The general solution:

$$y = -\frac{1}{2}\mathrm{e}^{-2t} - \frac{1}{4t}\mathrm{e}^{-2t} + \frac{C}{t}$$