# Poincare Classification HW Solutions

1. Use Poincare Classification to classify the origin in each case:

(a) $\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$ $\quad \mathrm{Tr}(A) = 4$, $\det(A) = 4$, $\Delta = 4^2 - 4 \cdot 4 = 0$ Degenerate Source

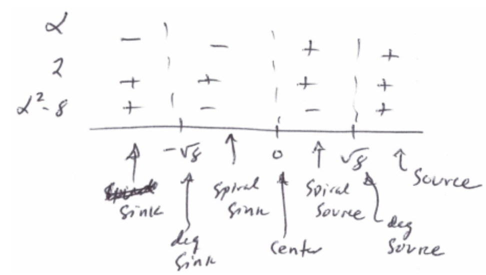
(b) $\begin{bmatrix} -1/2 & 1 \\ -1 & -1/2 \end{bmatrix}$ $\quad \mathrm{Tr}(A) = -1$, $\det(A) = 5/4$, $\Delta = -4 < 0$ Spiral Sink

(c) $\begin{bmatrix} -1 & -1 \\ 0 & -1/4 \end{bmatrix}$ $\quad \mathrm{Tr}(A) = -5/4$, $\det(A) = 1/4$, $\Delta = (25 - 16)/16 > 0$ Sink

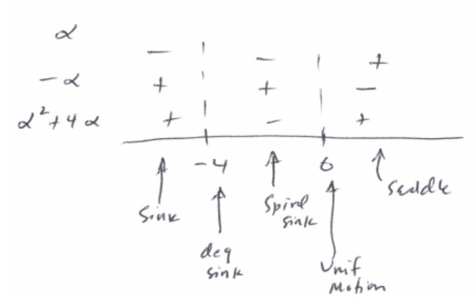
(d) $\begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$ $\quad \mathrm{Tr}(A) = 2$, $\det(A) = 5$, $\Delta = -16 < 0$ Spiral Source

2. For each matrix, find how the classification of the origin changes with $\alpha$:

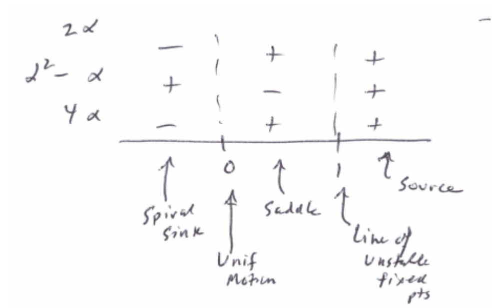
(a) $\begin{bmatrix} \alpha & -1 \\ 2 & 0 \end{bmatrix}$ $\quad \mathrm{Tr}(A) = \alpha$, $\det(A) = 2$, $\Delta = \alpha^2 - 8$



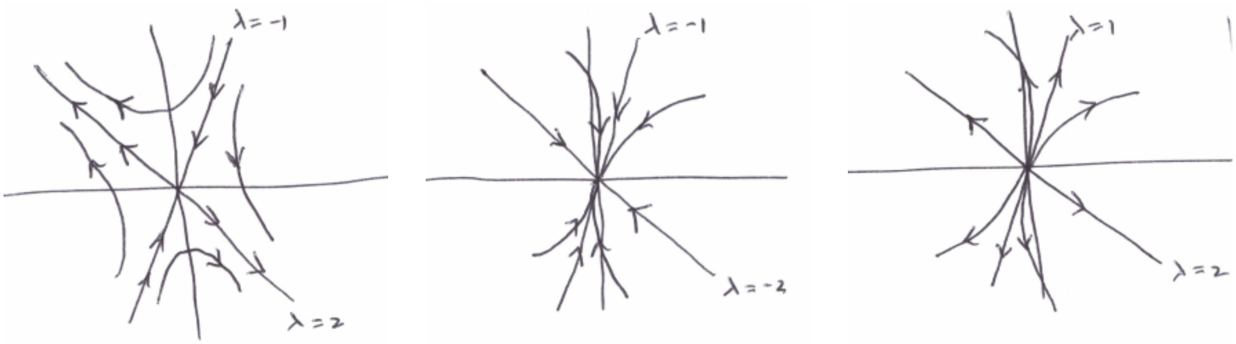

(b) $\begin{bmatrix} \alpha & \alpha \\ 1 & 0 \end{bmatrix}$ $\quad \mathrm{Tr}(A) = \alpha$, $\det(A) = -\alpha$, $\Delta = \alpha^2 + 4\alpha$




(c) $\begin{bmatrix} \alpha & 1 \\ \alpha & \alpha \end{bmatrix}$ $\quad \mathrm{Tr}(A) = 2\alpha$, $\det(A) = \alpha^2 - \alpha$, $\Delta = 4\alpha$



3. See the figures below.




4. (a) (Typo- Should be the same matrix as 1(d))

$$\begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} \quad \begin{array}{c} \mathrm{Tr}(A) = 2 \\ \det(A) = 5 \\ \lambda^2 - 2\lambda + 5 = 0 \end{array} \quad \begin{array}{c} \lambda = 1 + 2i \\ (3 - (1+2i))v_1 - 2v_2 = 0 \end{array}, \text{ so } \mathbf{v} = \begin{bmatrix} 1 \\ 1 - i \end{bmatrix}$$

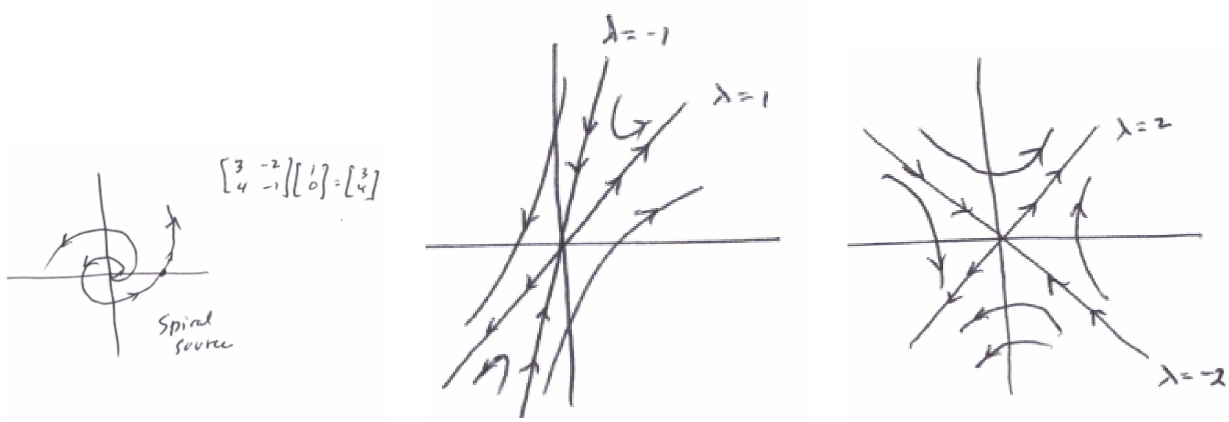
Now compute $\mathrm{e}^{\lambda t}\mathbf{v} = \mathrm{e}^{(1+2i)t} \begin{bmatrix} 1 \\ 1 - i \end{bmatrix}$, and we get:

$$\mathbf{x}(t) = \mathrm{e}^t \left( C_1 \begin{bmatrix} \cos(2t) \\ \cos(2t) + \sin(2t) \end{bmatrix} + C_2 \begin{bmatrix} \sin(2t) \\ -\cos(2t) + \sin(2t) \end{bmatrix} \right)$$

(b) $\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \quad \begin{array}{c} \mathrm{Tr}(A) = 0 \\ \det(A) = -1 \\ \lambda^2 - 1 = 0 \end{array}$ with $\mathbf{x}(t) = C_1 \mathrm{e}^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \mathrm{e}^{-t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \quad \begin{array}{c} \mathrm{Tr}(A) = 0 \\ \det(A) = -4 \\ \lambda^2 - 4 = 0 \end{array}$ with $\mathbf{x}(t) = C_1 \mathrm{e}^{-2t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + C_2 \mathrm{e}^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Figures for Problem 4:



# Homework solutions: 5-11 odd, 9.3

5. The critical points (equilibria) are found by first setting the first equation to zero, then go through each case for the second equation. In this case, if $x = -2$ in Equation 1, then $y = 2$ in the second. If $x = y$ in the first case, then either $x = 4$ or $x = 0$ in the second. This gives us the following equilibria:

$$(0,0), (4,0), \text{ and } (-2,2)$$

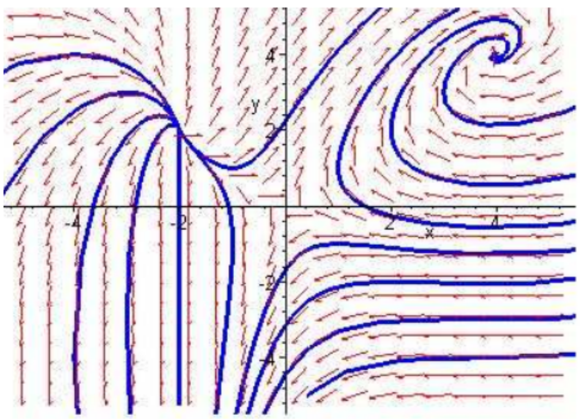
The Jacobian matrix is:

$$\begin{bmatrix} -2-2x+y & 2+x \\ 4-y-2x & 4-x \end{bmatrix}$$

At each of the equilibria (same order as above):

$$\begin{bmatrix} -2 & 2 \\ 4 & 4 \end{bmatrix} \begin{array}{c} \text{Tr}(A) = 2 \\ \det(A) = -16 \\ \Delta > 0 \end{array} \qquad \begin{bmatrix} -6 & 6 \\ -8 & 0 \end{bmatrix} \begin{array}{c} \text{Tr}(A) = -6 \\ \det(A) = 48 \\ \Delta < 0 \end{array} \qquad \begin{bmatrix} 4 & 0 \\ 6 & 6 \end{bmatrix} \begin{array}{c} \text{Tr}(A) = 10 \\ \det(A) = 24 \\ \Delta > 0 \end{array}$$

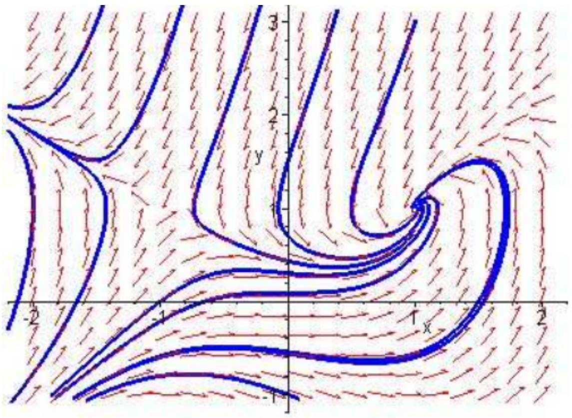
In order then, these are: Saddle, Spiral Sink, Source.




7. The equilibria are $(-1,1)$ and $(1,1)$. The Jacobian matrix is given by the following, with linearizations to follow (in the same order as above):

$$\begin{bmatrix} 0 & -1 \\ 2x & -2y \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 0 & -1 \\ -2 & -2 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 2 & -2 \end{bmatrix}$$

You should find that at $(-1,1)$, we have a saddle, and at $(1,1)$ we have a spiral sink.




9. In this case, we have 4 equilibria:

$$(0,0),(2,1),(2,-2),(4,-2)$$

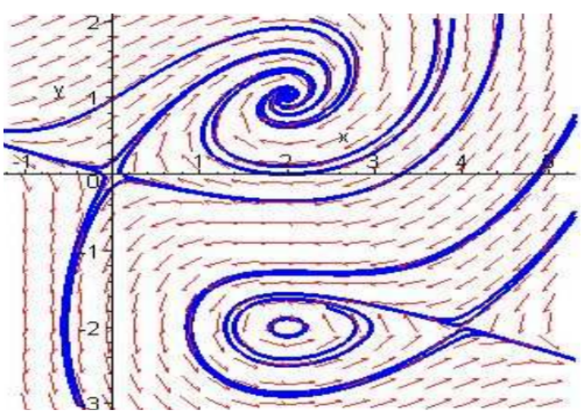
The Jacobian matrix is given by:

$$\begin{bmatrix} 1-y/2 & 2+2y-x/2 \\ -y+1-x & 2-x \end{bmatrix}$$

Evaluating at the four equilbria give, in order:

$$\begin{bmatrix} -1 & 2 \\ 1 & 2 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} -3/2 & 3 \\ -2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -3 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -4 \\ -1 & -2 \end{bmatrix}$$

You should find that these represent (in order),




11. Sorry about this one- The equilibria are not "nice", so this one should be done on a computer/calculator.