Poincare Classification

We solve $\mathbf{x}' = A\mathbf{x}$ by first computing the characteristic equation.

$$\lambda^2 - \text{Tr}(A)\lambda + \det(A) = 0 \qquad \lambda = \frac{\text{Tr}(A) \pm \sqrt{\Delta}}{2}$$

The type of solution depends on the discriminant on $\Delta = \text{Tr}^2(A) - 4\det(A)$. We looked at the cases where $\Delta > 0$, $\Delta = 0$ and $\Delta < 0$. For completeness, these are included below:

- $\Delta > 0$ means λ_1, λ_2 real, distinct.

$$\mathbf{x}(t) = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2$$

- $\Delta < 0$ means complex $\lambda = a + ib$, $\mathbf{v}$ (we only need one):

$$\mathbf{x}(t) = C_1 \text{Real}\left(e^{\lambda t}\mathbf{v}\right) + C_2 \text{Imag}\left(e^{\lambda t}\mathbf{v}\right)$$

- $\Delta = 0$ means a double eigenvalue. If in addition, there is only one eigenvector, then:

$$\mathbf{x}(t) = e^{\lambda t}\left(\mathbf{x}_0 + t\mathbf{w}\right)$$

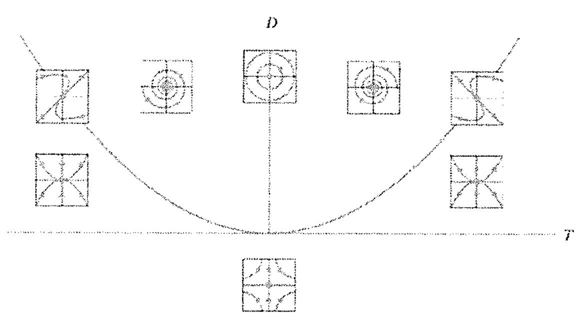
where $\mathbf{x}_0$ is the initial condition and $\mathbf{w}$ is any vector satisfying: $\begin{array}{ll} (a-\lambda)x_0 + by_0 & = w_1 \\ cx_0 + (d-\lambda)y_0 & = w_2 \end{array}$.

We see that $\Delta = 0$ separates the solutions, and if we examine this more carefully, we get:

$$\Delta = 0 \quad \Rightarrow \quad 0 = (\text{Tr}(A))^2 - 4\det(A)$$

This is a parabola in the $(\text{Tr}(A), \det(A))$ coordinate system.

This cuts the plane into several regions (you might also refer to Figure 9.1.9 on page 497).



Notice that inside the parabola (for example, where the trace is zero and the det. is positive) gives $\Delta < 0$ and outside the parabola (for example, where trace is zero and det. is negative) gives $\Delta > 0$.

We'll also consider a couple of specific regions to see how this diagram works. The first region is under the axis representing trace (so the determinant is negative). The second region is inside the parabola, but in the first quadrant.

1. In the first region the determinant is negative, and $\Delta > 0$. In examining λ,

$$\frac{\text{Tr}(A) \pm \sqrt{\text{Tr}(A)^2 - 4\det(A)}}{2}$$

   With det< 0, that means $(\text{Tr}(A))^2 < (\text{Tr}(A))^2 - 4\text{det}(A)$, so λ will ALWAYS be mixed in sign (pos, neg), and the origin will always be classified as a SADDLE.

2. In Quadrant I, the trace and determinant are both positive, and inside the parabola, $\Delta < 0$ (giving complex eigenvalues).

$$\lambda_{1,2} = \frac{\text{Tr}(A)}{2} \pm \frac{\sqrt{-\Delta}}{2} i$$

   Therefore, we have a SPIRAL SOURCE.

Here are some examples. Given the matrix A_i below, we will draw where we are on the Poincar'e Diagram, then classify the origin.

$$A_1 = \begin{bmatrix} 1 & 2 \\ -5 & -1 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \qquad A_3 = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$$

SOLUTIONS: We'll do these together in class.

Varying a Parameter

Suppose that the matrix has a parameter that varies. We can use the diagram to examine how changing the parameter changes the nature (or classification) of the origin.

For example, suppose the matrix, trace, determinant and discriminant are as shown:

$$A = \begin{bmatrix} a & a \\ 3 & -1 \end{bmatrix} \quad \Rightarrow \quad \begin{array}{c} \text{Tr}(A) = a - 1 \\ \det(A) = -a - 3a = -4a \\ \Delta = (a-1)^2 - 4(-4a) = (a-1)^2 + 16a \end{array}$$

Homework: Poincare Classification

1. The matrices below represent the matrix A in a linear system, $\mathbf{x}' = A\mathbf{x}$. Classify the equilibrium (the origin) using the Poincaré Classification:

   (a) $\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$

   (b) $\begin{bmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{bmatrix}$

   (c) $\begin{bmatrix} -1 & -1 \\ 0 & -\frac{1}{4} \end{bmatrix}$

   (d) $\begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$

2. Discuss how the classification of the origin changes with α, given the matrix below.

   (a) $\begin{bmatrix} \alpha & -1 \\ 2 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} \alpha & \alpha \\ 1 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} \alpha & 1 \\ \alpha & \alpha \end{bmatrix}$

3. Suppose we are given $\mathbf{x}' = A\mathbf{x}$, and we compute the eigenvalues and eigenvectors (below). Draw a sketch of the phase plane in each case.

   (a) $\lambda_1 = -1, \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ with $\lambda_2 = 2, \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

   (b) $\lambda_1 = -1, \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ with $\lambda_2 = -2, \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

   (c) $\lambda_1 = 1, \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ with $\lambda_2 = 2, \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

4. Each matrix below represents the matrix A in the system $\mathbf{x}' = A\mathbf{x}$. For each, (i) Use the Poincaré classification, (ii) write down the general solution using eigenvalues/eigenvectors, and (iii) draw a sketch of the phase plane (the (x_1, x_2) plane).

   (a) $\begin{bmatrix} 3 & -2 \\ 4 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$

Local Linear Analysis of Nonlinear Autonomous DEs

In Calculus III, the *linearization* of $z = f(x,y)$ at a point $x = a, y = b$ was defined to be the tangent plane, whose equation is given by:

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)$$

Local Linearization

Local linear analysis is the process by which we analyze a nonlinear system of differential equations by linearizing that system near its equilibrium solutions.

Given a system of nonlinear autonomous DEs:

$$\begin{aligned} x' &= f(x,y) \\ y' &= g(x,y) \end{aligned} \qquad \text{or} \qquad \mathbf{x}'(t) = F(\mathbf{x})$$

we first find the equilibrium solutions by setting the derivatives to zero, then solve simultaneously, the system:

$$\begin{aligned} f(x,y) &= 0 \\ g(x,y) &= 0 \end{aligned}$$

Given an equilibrium, say $x = a, y = b$, the linearization of the system at the point (a,b) is the following matrix, also called the **Jacobian matrix** of F.

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} f_x(a,b) & f_y(a,b) \\ g_x(a,b) & g_y(a,b) \end{bmatrix} \begin{bmatrix} x-a \\ y-b \end{bmatrix}$$

We can then use the Poincaré Diagram to determine the local behavior. We must use some caution in the case of centers and degenerate nodes, however. Because the linearization is an *approximation* of the true solution, the actual solutions are of a slightly perturbed system. This means that while the linearization gives a center, the true solution may be a center or a spiral (we would use a computer simulation to see what we actually get).

Example 1: Competing Species

Suppose we have two populations that are competing for similar resources, like rabbits ($x(t)$) and hamsters ($y(t)$).

It seems reasonable to suppose that in the absence of the other,, each population is modeled by the logistic equation (population model with a threshold). In that case, assuming the constants are known, we might have something like the following:

$$\begin{aligned} x' &= x - x^2 \\ y' &= \tfrac{3}{4}y - y^2 \end{aligned}.$$

The second assumption will be that, because the two species are competing for resources, then the rate of change of both populations will decrease in the presence of the other. We'll assume that it is proportional to the number of interactions between the populations. Given a couple of constants for those rates, our equations might be something like this:

$$\begin{aligned} x' &= x - x^2 - xy \\ y' &= \tfrac{3}{4}y - y^2 - \tfrac{1}{2}xy \end{aligned}.$$

Finding Equilibria

Before we go to solve directly for the equilibria, we'll define a new term: The **nullclines** for a system are the curves for which one of the derivatives is zero.

In the competing species model, from the first equation, we have two nullclines-

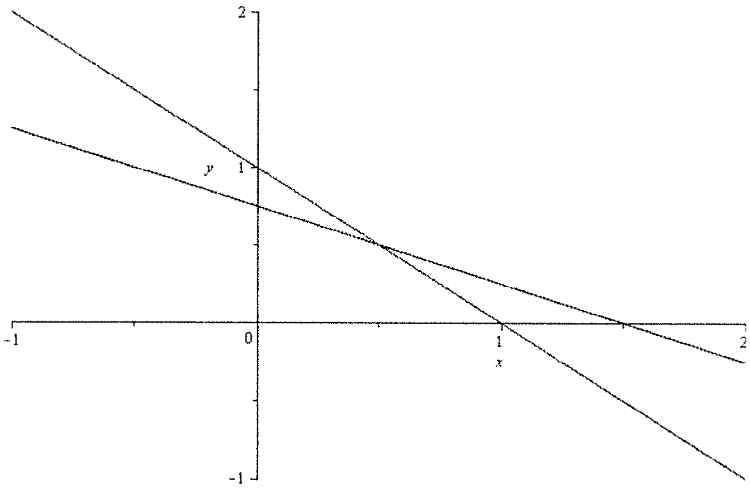
$$x(1-x-y) = 0 \quad \Rightarrow \quad x = 0 \quad \text{or } y = 1-x$$

So the vertical line $x = 0$ and the line $y = 1-x$ are curves in the plane where $x' = 0$.

Similarly, from the second equation:

$$y\left(\frac{3}{4} - y - \frac{1}{2}x\right) = 0 \quad \Rightarrow \quad \text{or } y = -\frac{1}{2}x + \frac{3}{4}$$

And so the horizontal line $y = 0$ and the other line are curves where $y' = 0$. If we draw all these lines, you might notice that the equilibria are points where the nullclines from different equations meet. In the diagram below, we might be able to read off the equilibria, although we give a second method for finding them below.



To get the equilibria analytically (rather than from the nullclines), set each derivative to zero. From the first equation, either $x = 0$ or $x = -y + 1$:

- If $x = 0$ then the second equation: $y\left(\frac{3}{4} - y\right) = 0$ so that $y = 0$ or $y = 3/4$.
- If $x = -y + 1$ then the second equation becomes:

$$y\left(\frac{3}{4} - y - \frac{1}{2}(-y + 1)\right) = y\left(\frac{1}{4} - \frac{1}{2}y\right) = 0$$

Therefore, $y = 0$ (and $x = 0$, but we've counted that one), or $y = 1/2$ (then $x = 1/2$, too).

We have 4 equilibrium solutions:

$$(0, 0),\ (1, 0),\ (0, 3/4),\ (1/2, 1/2)$$

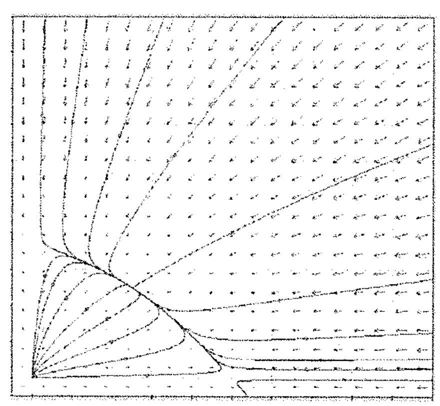
Now we linearize the system about each equilibrium solution to determine its stability. First, the matrix of partial derivatives is:

$$\begin{bmatrix} f_x(a,b) & f_y(a,b) \\ g_x(a,b) & g_y(a,b) \end{bmatrix} = \begin{bmatrix} 1 - 2x - y & -x \\ -0.5y & 0.75 - 2y - 0.5x \end{bmatrix}$$

Evaluating this at each of the equilibrium (in order) gives us:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0.75 \end{bmatrix} \quad \begin{bmatrix} -1 & -1 \\ 0 & 0.25 \end{bmatrix} \quad \begin{bmatrix} 0.25 & 0 \\ -.375 & -.75 \end{bmatrix} \quad \begin{bmatrix} -.5 & -.5 \\ -.25 & -.5 \end{bmatrix}$$

Using the Poincaré Diagram, we see that the origin is a SOURCE, the equilibria on the $x-$ and $y-$ axes are SADDLES, and the point of intersection of the two lines is a SINK. Putting these together, we can look at the direction field to examine the global behavior. From this we see that if both of the initial populations are not zero, the model predicts that all solutions will tend to the sink at $(1/2, 1/2)$- We might call this **peaceful coexistence**.



Example 2: Predator-Prey

Consider the following example of a predator-prey system:

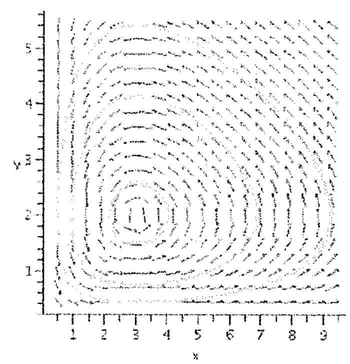
$$\begin{aligned} x' &= x - 0.5xy &&= x(1-0.5y) \\ y' &= -0.75y + 0.25xy &&= y(-0.75+0.25x) \end{aligned}$$

Analyze the behavior of the solutions of the system by using local linearization.

SOLUTION: You might plot the nullclines first to get a sense for where the equilibria will occur. Solving for the equilibria, you should get only two: $(0,0)$ and $(3,2)$.

Now compute the "Jacobian matrix" of partial derivatives:

$$\begin{bmatrix} f_x(a,b) & f_y(a,b) \\ g_x(a,b) & g_y(a,b) \end{bmatrix} \Rightarrow \begin{bmatrix} 1-0.5y & -0.5x \\ 0.25y & -0.75+0.25x \end{bmatrix}$$



Linearizing about the two equilibrium gives (in order):

$$\begin{bmatrix} 1 & 0 \\ 0 & -0.75 \end{bmatrix} \quad \begin{bmatrix} 0 & -1.5 \\ 0.5 & 0 \end{bmatrix}$$

Using the Poincare Diagram, we should see that the origin is a saddle, and $(3,2)$ is a center.

Homework

For homework, use the following systems of equations, but instead of the listed instructions, do: (i) Determine all equilibrium solutions (equilibrium solutions are the same as critical points), (ii) find the corresponding linear system near each equilibrium, and (iii) use the Poincare classification diagram to see what local behavior to expect.

The systems are from section 9.3, exercises 5-11 odd.