Using the Complex Exponential

In the Method of Undetermined coefficients, if the forcing function is an exponential function, then our guess is straightforward. We guess that the particular function is a constant times the exponential, differentiate that twice, and substitute into the DE to find the constant.

For example, given $y'' + 2y' + y = \mathrm{e}^{2t}$, the homogeneous part of the solution has $r = -1, -1$, so that

$$y_p = A\mathrm{e}^{2t} \quad \Rightarrow \quad y_p' = 2A\mathrm{e}^{2t} \quad \Rightarrow \quad y_p'' = 4A\mathrm{e}^{2t}$$

and:

$$A\mathrm{e}^{2t}(4 + 2(2) + 1) = \mathrm{e}^{2t} \quad \Rightarrow \quad A = \frac{1}{9}$$

Therefore, $y_p = \frac{1}{9}\mathrm{e}^{2t}$

In class, we said that if the forcing function was either $\cos(at)$ or $\sin(at)$, then our ansatz would be: $y_p = A\cos(at) + B\sin(at)$. We then differentiate twice and substitute.

However, if we're comfortable with the complex numbers and the complex exponential, then we can use a shortcut. Here are two related examples.

Example 1

Solve

$$y'' + 2y' + y = \cos(3t)$$

SOLUTION: Rather than solve this problem, we will solve a related problem by changing the right side of the equation into an exponential function!

$$y'' + 2y' + y = \cos(3t) + i\sin(3t) = \mathrm{e}^{3it}$$

We don't need the full solution- Because $\cos(3t)$ is the real part of the exponential, we will solve the DE and only use the real part as our answer. In doing this, we've changed the forcing function from "trig" type to "exponential type", so our ansatz is now $y_p = A\mathrm{e}^{3it}$.

Continuing, treat the forcing function like a regular exponential and solve for the unknown constant A:

$$y_p = A\mathrm{e}^{3it} \qquad y_p' = 3iA\mathrm{e}^{3it} \qquad y_p'' = -9A\mathrm{e}^{3it}.$$

Substituting into the DE, we get:

$$A\mathrm{e}^{3it}(-9 + 2(3i) + 1) = \mathrm{e}^{3it} \quad \Rightarrow \quad A = \frac{1}{-8+6i} = \frac{1}{2}\,\frac{1}{-4+3i}$$

where I've factored out the 1/2 to make the computations easier.

Our solution is the real part of $A\mathrm{e}^{3it}$, so we compute that to get our answer. Leaving off the 1/2,

$$\frac{1}{-4+3i}\left(\cos(3t) + i\sin(3t)\right) = \frac{-4-3i}{25}\left(\cos(3t) + i\sin(3t)\right)$$

If we multiply this out, we get:

$$\left(-\frac{4}{25}\cos(3t) + \frac{3}{25}\sin(3t)\right) + i\left(-\frac{3}{25}\cos(3t) - \frac{4}{25}\sin(3t)\right)$$

We now have our answer- Note that we did not actually need to compute the imaginary part, we did so in this case to show the complete computation, if you needed it. In our case, we only need the real part, so the particular solution is the following (remember to put the 1/2 back in)

$$y_p = \mathrm{Real}(A\mathrm{e}^{3it}) = -\frac{4}{50}\cos(3t) + \frac{3}{50}\sin(3t)$$

Example 2

Solve: $y'' + 2y' + y = \sin(3t)$

In this case, we would use exactly the same technique as before, but we go after the imaginary part of the solution at the end. That is, we:

- Replace $\sin(3t)$ by e^{3it}
- Use $y_p = A\mathrm{e}^{3it}$.
- Take the imaginary part of y_p as the solution.

And we can see what our solution is:

$$y_p = -\frac{3}{50}\cos(3t) - \frac{4}{50}\sin(3t)$$

Example 3

Solve: $y'' + 9y = \sin(3t)$.

If we were to do this using sine and cosine, we would have to guess:

$$y_p = t(A\cos(3t) + B\sin(3t))$$

But wouldn't it be a bit easier to solve using the complex exponential? In that case, we solve the following, again noting that the homogeneous part of the solution is

$$y_h = C_1\cos(3t) + C_2\sin(3t)$$

We re-write the ODE as:

$$y'' + 9y = \mathrm{e}^{3it}$$

And take the following as our ansatz (multiplied by t). Recall that we'll actually only need the imaginary part of y_p:

$$y_p = At\mathrm{e}^{3it} \qquad y_p' = A\mathrm{e}^{3it} + 3iAt\mathrm{e}^{3it} \qquad y_p'' = 6iA\mathrm{e}^{3it} - 9At\mathrm{e}^{3it}$$

Now,

$$y_p'' + 9y_p = 6iA\mathrm{e}^{3it} \quad \Rightarrow \quad 6iA\mathrm{e}^{3it} = \mathrm{e}^{3it}.$$

From this, we see that:

$$A = \frac{1}{6i} = -\frac{1}{6}i$$

As before, we want the imaginary part of $A\mathrm{e}^{3it}$, which in this case will be:

$$y_p = \mathrm{Imag}(At\mathrm{e}^{3it}) = \mathrm{Imag}\left(-\frac{t}{6}\,i(\cos(3t) + i\sin(3t)\right) = \frac{t}{6}\sin(3t)$$

Example 4

Let's try another, where we will assume the right side of the equation is the cosine.

$$y'' + y' - 2y = \cos(2t)$$

Now, using our trick of embedding this problem into a larger problem:

$$y'' + y' - 2y = \cos(2t) + i\sin(2t) = \mathrm{e}^{2it}$$

The ansatz and its derivatives:

$$y_p = A\mathrm{e}^{2it} \qquad y_p' = 2iA\mathrm{e}^{2it} \qquad y_p'' = -4A\mathrm{e}^{2it}$$

Putting these into the DE and solve for A:

$$A\mathrm{e}^{2it}\left(-4 + 2i - 2(1)\right) = \mathrm{e}^{2it} \quad \Rightarrow \quad A = \frac{1}{-6+2i}$$

Then find the real part of $A\mathrm{e}^{2it}$:

$$\text{Real}\left(\frac{-6-2i}{40}\left(\cos(2t) + i\sin(2t)\right)\right) = -\frac{6}{40}\cos(2t) + \frac{2}{40}\sin(2t)$$

I've left the fractions unsimplified so that you can see what was done.

Example 5

The complex exponential can also be used to help compute integrals involving the sine and cosine. As a simple example:

$$\int \cos(3t)\,dt = \text{Real}\left(\int \mathrm{e}^{3it}\,dt\right) = \text{Real}\left(\frac{1}{3i}\mathrm{e}^{3it}\right) = \frac{1}{3}\sin(3t) + C$$

We can do the same with some harder integral- Namely, an exponential times sine or cosine. We'll embed this into a larger problem:

$$\int \mathrm{e}^{-t}\sin(2t)\,dt \quad \Rightarrow \quad \int \mathrm{e}^{-t}(\cos(2t) + i\sin(2t))\,dt = \int \mathrm{e}^{-t}\mathrm{e}^{2it}\,dt = \int \mathrm{e}^{(-1+2i)t}\,dt$$

And the exponential is easy to integrate. The solution is the imaginary part of:

$$\frac{1}{-1+2i}\mathrm{e}^{2it} = \frac{-1-2i}{5}(\cos(2t) + i\sin(2t))$$

Therefore,

$$\int \mathrm{e}^{-t}\sin(2t)\,dt = -\frac{2}{5}\cos(2t) - \frac{1}{5}\sin(2t) + C$$

We'll take a look at one more example that will be very useful in Section 3.8.

Section 3.8 Shortcut

Suppose we have the DE:

$$y'' + py' + qy = \cos(omegat)$$

where we assume p, q are real numbers. If we use our shortcut assumption, we can solve for the constant A coming from the Method of Undetermined Coefficients:

$$y_p = A\mathrm{e}^{i\omega t} \quad \Rightarrow \quad A\mathrm{e}^{i\omega t}(-\omega^2 + i\omega p + q) = \mathrm{e}^{i\omega t}$$

which leaves us with:

$$A = \frac{1}{(q-\omega^2) + i\omega p} = \frac{1}{\alpha + i\beta}$$

We will show that, if we write $y_p = R\cos(\omega t - \delta)$, then

$$R = \frac{1}{|\alpha + i\beta|} \qquad \text{and} \qquad \delta = \arg(\alpha + \beta i) = \arctan\left(\frac{\beta}{\alpha}\right)$$

Proof: Compute the real part:

$$\text{Real}\left(\frac{\alpha-\beta i}{\alpha^2+\beta^2}(\cos(\omega t)+i\sin(\omega t)\right)=\frac{\alpha}{\alpha^2+\beta^2}\cos(\omega t)+\frac{\beta}{\alpha^2+\beta^2}\sin(\omega t)=R\cos(\omega t-\delta)$$

with

$$R=\sqrt{\frac{\alpha^2}{(\alpha^2+\beta^2)^2}+\frac{\beta^2}{(\alpha^2+\beta^2)^2}}=\frac{1}{\sqrt{\alpha^2+\beta^2}}=\frac{1}{|\alpha+\beta i|}$$

And for the phase angle δ, the sum of squares terms cancel out leaving us with the angle for $\alpha+\beta i$, or

$$\delta=\arctan\left(\frac{\beta}{\alpha}\right)$$

Using our previous results...

Find y_p if

$$y''+2y'+y=\cos(2t)$$

SOLUTION: Let $y_p=A\mathrm{e}^{2it}$ and substitute:

$$A\mathrm{e}^{2it}\left(-4+4i+1\right)=\mathrm{e}^{2it}\qquad\Rightarrow\qquad A=\frac{1}{-3+4i}$$

Therefore, the amplitude and phase shift for y_p is:

$$R=\frac{1}{\sqrt{9+16}}=\frac{1}{5}\qquad\delta=\tan^{-1}(-4/3)+\pi$$

and $y_p=R\cos(2t-\delta)$.

We see that, given a cosine forcing function allows us to very quickly get the "forced response", y_p.

Practice:

1. Use the complex exponential to integrate the following:
   (a) $\displaystyle\int \mathrm{e}^{-2t}\cos(t)\,dt$
   (b) $\displaystyle\int \mathrm{e}^{t/2}\sin(3t)\,dt$
2. Use the complex exponential to find y_p, given:
   (a) $y''+7y=3\cos(3t)$
   (b) $y''+y'+3y=2\sin(2t)$
   (c) $y''+2y'+y=\cos(2t)$
3. Use the complex exponential to find the amplitude and phase angle for the forced response:

$$y''+2y'+2y=\cos(t)$$

Solutions to the Practice

1. Use the complex exponential to integrate the following:

   (a) $\int e^{-2t}\cos(t)\,dt$
   SOLUTION: We'll take the real part of:

   $$\int e^{-2t}e^{it}\,dt = \int e^{(-2+i)t}\,dt = \frac{1}{-2+i}e^{(-2+i)t}$$

   Expanding this,

   $$\text{Real}\left(e^{-2t}\left(\frac{-2-i}{5}(\cos(t)+i\sin(t))\right)\right) = e^{-2t}\left(-\frac{2}{5}\cos(t)+\frac{1}{5}\sin(2t)\right)+C$$

   (b) $\int e^{t/2}\sin(3t)\,dt$
   SOLUTION: Same idea as before:

   $$\int e^{t/2}e^{3it}\,dt = \int e^{\left(\frac{1}{2}+3i\right)t}\,dt = \frac{1}{\frac{1}{2}+3i}e^{\left(\frac{1}{2}+3i\right)t} = \frac{2}{1+6i}e^{t/2}(\cos(3t)+i\sin(3t))$$

   A little more simplification:

   $$2e^{t/2}\left(\frac{-1-6i}{37}(\cos(3t)+i\sin(3t)\right)$$

   And we take the imaginary part, so that:

   $$\int e^{t/2}\sin(3t)\,dt = e^{t/2}\left(-\frac{12}{37}\cos(3t)+\frac{2}{37}\sin(3t)\right)+C$$

2. Use the complex exponential to find y_p, given:

   (a) $y''+7y=3\cos(3t)$
   SOLUTION: The ansatz for $y''+7y=3e^{3it}$ would be $y_p=Ae^{3it}$ (and we'll keep the real part). Substitute into the DE and factor out Ae^{3it}:

   $$Ae^{3it}(-9+7)=3e^{3it} \quad\Rightarrow\quad A=\frac{-3}{2}$$

   The particular part of the solution is $y_p=-\frac{3}{2}\cos(3t)$.

   (b) $y''+y'+3y=2\sin(2t)$
   SOLUTION: The ansatz for $y''+y'+3y=2\sin(2t)$ is $y_p=Ae^{2it}$ (imaginary part). Substitute:

   $$Ae^{2it}(-4+2i+3)e^{3it}=2e^{3it} \quad\Rightarrow\quad A=\frac{2}{-1+2i}$$

   We take the imaginary part of the expression below (the complex number has been rationalized):

   $$\frac{-2-4i}{5}(\cos(2t)+i\sin(2t)) \quad\Rightarrow\quad y_p=-\frac{2}{5}\cos(2t)-\frac{2}{5}\sin(2t)$$

(c) $y'' + 2y' + y = \cos(2t)$

SOLUTION: Not quite as easy as before, although not bad. We take $y'' + 2y' + y = \mathrm{e}^{2it}$, and $y_p = A\mathrm{e}^{2it}$. Substitute and solve:

$$A\mathrm{e}^{2it}(-4 + 2(2i) + 1) = \mathrm{e}^{2it} \quad \Rightarrow \quad A = \frac{1}{-3+4i}$$

We want the real part of the product:

$$\frac{1}{-3+4i}(\cos(2t) + i\sin(2t)) = \frac{-3-4i}{25}(\cos(2t) + i\sin(2t))$$

which is: $y_p = -\frac{3}{25}\cos(2t) + \frac{4}{25}\sin(2t)$

3. Use the complex exponential to find the amplitude and phase angle for the forced response:

$$y'' + 2y' + 2y = \cos(t)$$

For this one, we can use our shortcut. Take the ODE to be: $y''+2y'+2y = \mathrm{e}^{it}$, so $y_p = A\mathrm{e}^{it}$. Substitute into the DE to get:

$$A\mathrm{e}^{it} = (-1 + 2i + 2) = \mathrm{e}^{it} \quad \Rightarrow \quad A = \frac{1}{1+2i}$$

The amplitude and phase shift are:

$$R = \frac{1}{|1+2i|} = \frac{1}{5} \qquad \delta = \tan^{-1}(2)$$