Lecture Notes to substitute for 7.3-7.5

We want to solve the system:

$$\begin{aligned} x_1' &= ax_1 + bx_2 \\ x_2' &= cx_1 + dx_2 \end{aligned} \quad \leftrightarrow \quad \mathbf{x}' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mathbf{x} \quad \leftrightarrow \quad \mathbf{x}' = A\mathbf{x}$$

SOLUTION: Use the ansatz $\mathbf{x}(t) = \mathrm{e}^{\lambda t}\mathbf{v}$, then $\mathbf{x}' = \lambda \mathrm{e}^{\lambda t}\mathbf{v}$, so that the DE becomes:

$$A\mathrm{e}^{\lambda t}\mathbf{v} = \lambda \mathrm{e}^{\lambda t}\mathbf{v} \quad \Rightarrow A\mathbf{v} = \lambda\mathbf{v} \quad \text{or} \quad \begin{aligned} av_1 + bv_2 &= \lambda v_1 \\ cv_2 + dv_2 &= \lambda v_2 \end{aligned}$$

If the system above is true for that particular value of λ and **non-zero** vector $\mathbf{v}$, then λ is an **eigenvalue** of the matrix A and $\mathbf{v}$ is an associated **eigenvector**. Note that while $\mathbf{v}$ is not allowed to be the zero vector, λ could be zero.

Computing λ and $\mathbf{v}$:

We showed that in order for $\mathrm{e}^{\lambda t}\mathbf{v}$ to solve the system, we must have:

$$\begin{aligned} av_1 + bv_2 &= \lambda v_1 \\ cv_1 + dv_2 &= \lambda v_2 \end{aligned} \quad \Rightarrow \quad \begin{aligned} (a-\lambda)v_1 \quad &+ bv_2 = 0 \\ cv_1 \quad &+ (d-\lambda)v_2 = 0 \end{aligned}$$

That is the key system of equations. We saw last time $A\mathbf{x} = 0$ has exactly the zero solution iff $\det(A) \neq 0$. Therefore, for this system to have a **non-trivial solution** (which is a non-zero eigenvector), the **determinant must be zero.**

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad \lambda^2 - (a+d)\lambda + (ad-bc) = 0$$

You might recognize those two quantities that are computed as the trace and determinant of A:

$$\mathrm{Tr}(A) = a + d \qquad \det(A) = ad - bc$$

Theorem: The eigenvalues for the 2×2 matrix A are found by solving the **characteristic equation**:

$$\lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = 0$$

We could solve using the quadratic formula. Just as in Chapter 3, the form of the solution will depend on whether the discriminant is positive (two real λ), negative (two complex λ) or zero (one real λ). Today, we will **focus on the distinct eigenvalues** case.

Theorem: If $\lambda_1, \mathbf{v}_1$ and $\lambda_2, \mathbf{v}_2$ are the (real, distinct) eigenvalues and eigenvectors for our system, then the general solution to the differential equation is given by:

$$\mathbf{x}(t) = C_1\mathrm{e}^{\lambda_1 t}\mathbf{v}_1 + C_2\mathrm{e}^{\lambda_2 t}\mathbf{v}_2$$

Example 1:

Find the eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$.

SOLUTION: We could jump right to the characteristic equation, but for practice its good to write down what it is we actually want to solve (the unknowns below are λ, v_1, v_2):

$$\begin{array}{rl} 7v_1 + 2v_2 & = \lambda v_1 \\ -4v_1 + v_2 & = \lambda v_2 \end{array} \quad \Rightarrow \quad \begin{array}{rl} (7-\lambda)v_1 + 2v_2 & = 0 \\ -4v_2 + (1-\lambda)v_2 & = 0 \end{array}$$

For this to have a non-zero solution v_1, v_2, the determinant must be zero:

$$(7-\lambda)(1-\lambda) + 8 = 0 \quad \Rightarrow \quad \lambda^2 - 8\lambda + 15 = 0$$

This factors, so we can solve for λ:

$$(\lambda - 5)(\lambda - 3) = 0 \quad \Rightarrow \quad \lambda = 3, 5$$

Now, for each λ, go back to our system of equations for v_1, v_2 and solve. NOTE: By design, these equations should be multiples of each other!

For $\lambda = 3$:

$$\begin{array}{rl} (7-3)v_1 + 2v_2 & = 0 \\ -4v_2 + (1-3)v_2 & = 0 \end{array} \quad \Rightarrow \quad 4v_1 + 2v_2 = 0$$

There are an infinite number of solutions (and there always be). We want to choose "nice" values of v_1, v_2 that satisfies this relationship (or alternatively, lies on the line). An easy way of writing v_1, v_2 is to notice the following:

Given $ax + by = 0$, we can choose $x = b, y = -a$ to lie on the line.

Continuing, we'll take our vector $\mathbf{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$. Now go through the same process to find the eigenvector for $\lambda = 5$:

$$\begin{array}{rl} (7-5)v_1 + 2v_2 & = 0 \\ -4v_1 + (1-5)v_2 & = 0 \end{array} \quad \Rightarrow \quad 2v_1 + 2v_2 = 0 \quad \Rightarrow \quad v_1 + v_2 = 0 \quad \Rightarrow \quad \mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

You might note: If $\mathbf{v}$ is an eigenvector, then so is any scalar multiple of $\mathbf{v}$ (that is, the set of all eigenvectors forms a line). Therefore, when computing eigenvectors by hand, we typically re-scale them so that they are integers.

Example 2:

Solve: $\mathbf{x}' = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \mathbf{x}$

SOLUTION: First compute the eigenvalues and eigenvectors. The determinant is 8, the trace is 6. The characteristic equation is:

$$\lambda^2 - 6\lambda + 8 = 0 \quad \Rightarrow \quad (\lambda - 2)(\lambda - 4) = 0$$

Therefore, $\lambda = 2, 4$.

- For $\lambda = 2$:

$$\begin{aligned} (3-2)v_1 + v_2 &= 0 \\ v_1 + (3-2)v_2 &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} v_1 + v_2 &= 0 \\ v_1 + v_2 &= 0 \end{aligned} \quad \Rightarrow \quad \mathbf{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- For $\lambda = 4$,

$$\begin{aligned} (3-4)v_1 + v_2 &= 0 \\ v_1 + (3-4)v_2 &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} -v_1 + v_2 &= 0 \\ v_1 - v_2 &= 0 \end{aligned} \quad \Rightarrow \quad \mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The general solution is:

$$\mathbf{x}(t) = C_1 e^{2t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + C_2 e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Example 3:

Solve $\mathbf{x}' = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \mathbf{x}$.

SOLUTION: $\mathrm{Tr}(A) = 1$, the $\det(A) = -2$ so the characteristic equation is $\lambda^2 - \lambda - 2 = 0$, or $(\lambda + 1)(\lambda - 2) = 0$.

The eigenvalues are $\lambda = -1, 2$. The corresponding eigenvectors are found by solving the system above. For $\lambda = -1$:

$$\begin{aligned} (3+1)v_1 - 2v_2 &= 0 \\ 2v_1 + (-2+1)v_2 &= 0 \end{aligned} \qquad 2v_1 - v_2 = 0 \qquad \mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

For $\lambda = 2$:

$$\begin{aligned} (3-2)v_1 - 2v_2 &= 0 \\ 2v_1 + (-2-2)v_2 &= 0 \end{aligned} \qquad v_1 - 2v_2 = 0 \qquad \mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The solution to the system of differential equations is then:

$$\mathbf{x}(t) = C_1 e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Visualizing the Solutions

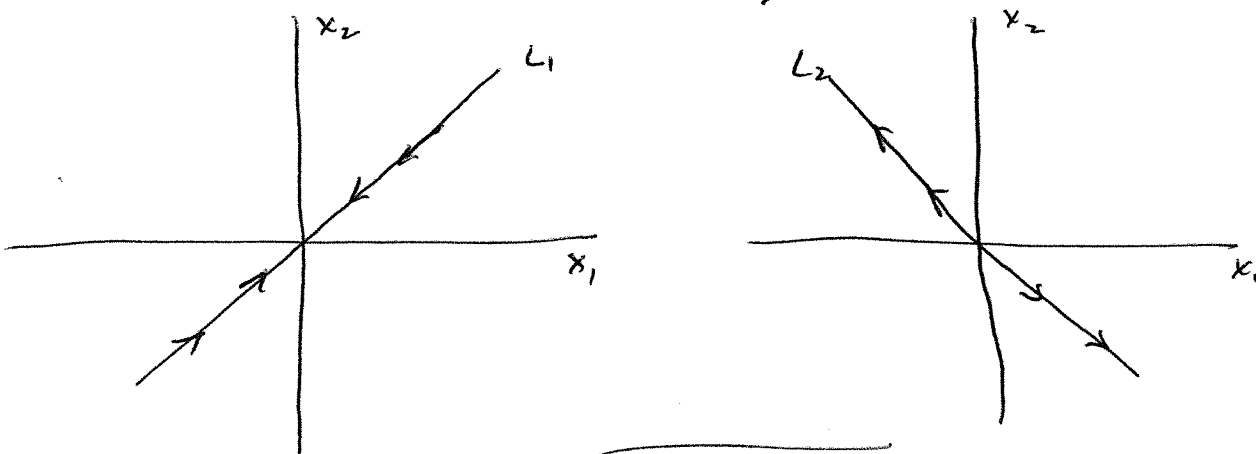
Given a function in time like $Ce^{\lambda t}\mathbf{v}$, varying time only means that we have different scalar multiples of the vector $\mathbf{v}$. Since we have two eigenvalues/eigenvectors, we'll have two lines:

$$L_1 = C_1 e^{\lambda_1 t} \mathbf{v}_1 \qquad L_2 = C_2 e^{\lambda_2 t} \mathbf{v}_2$$

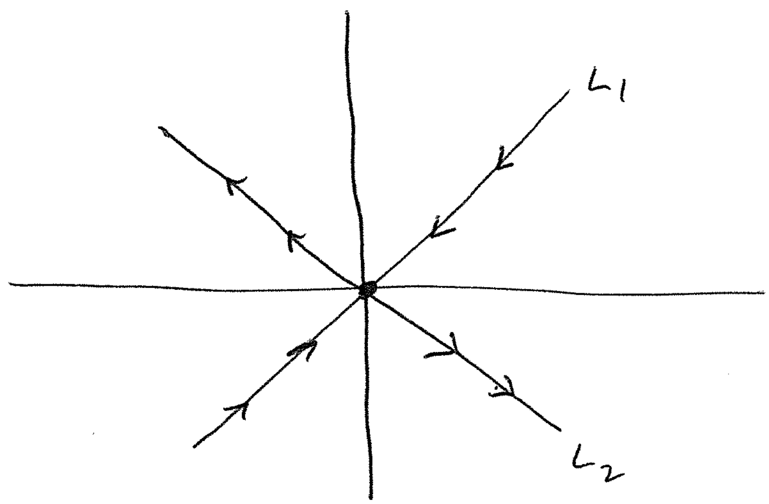
The full solution is made from adding these together for specific constants C_1, C_2.

Sketch the solutions to:

$$\underbrace{c_1 e^{-t}\begin{bmatrix}1\\1\end{bmatrix}}_{L_1} + \underbrace{c_2 e^{t}\begin{bmatrix}1\\-1\end{bmatrix}}_{L_2}$$



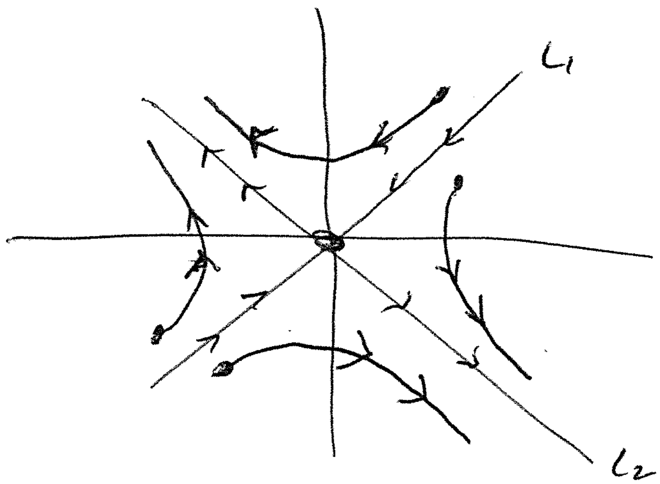
Put these together.



For other solutions, for example:

$$\underbrace{\tfrac{1}{2} e^{-t}}_{\to 0}\begin{bmatrix}1\\1\end{bmatrix} + \underbrace{\tfrac{3}{10} e^{t}}_{\to \infty}\begin{bmatrix}1\\-1\end{bmatrix}$$

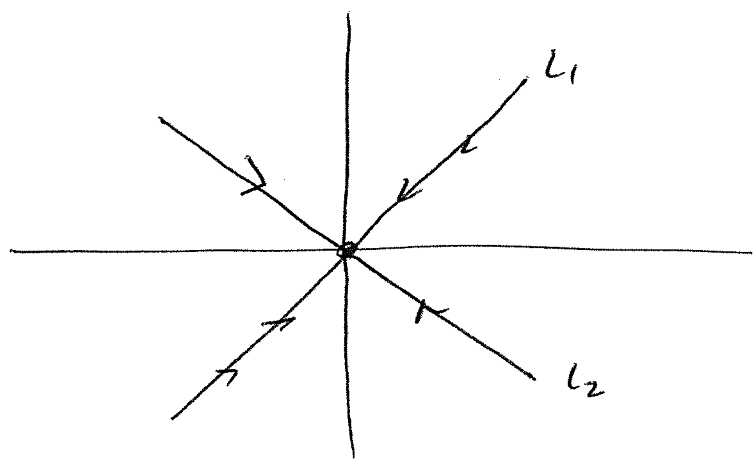
Solutions go to infinity along line L_2.



Solutions move to "∞" along line L_2

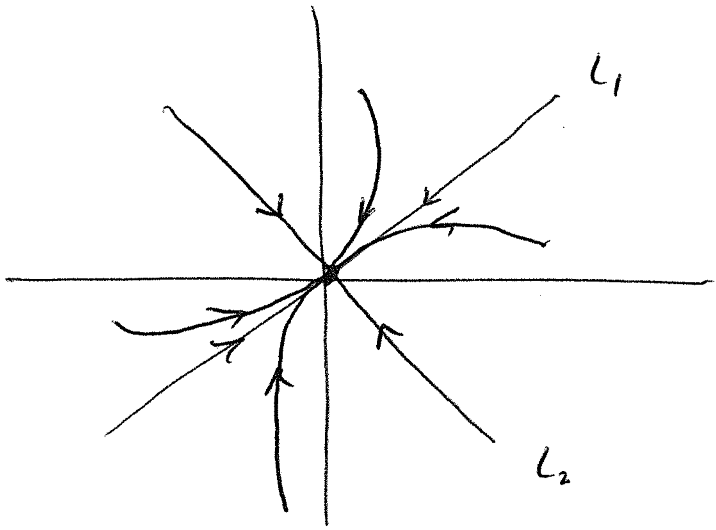
(This is an example of a "saddle").

Example: $c_1 e^{-t}\begin{bmatrix}1\\1\end{bmatrix} + c_2 e^{-2t}\begin{bmatrix}1\\-1\end{bmatrix}$



All soln's converge to the origin.

Note that $c_2 e^{-2t}\begin{bmatrix}1\\-1\end{bmatrix}$ will go to zero faster than $c_1 e^{-t}\begin{bmatrix}1\\1\end{bmatrix}$, so $\vec{x}(t)$ will approach the origin tangent to $c_1 e^{-t}\begin{bmatrix}1\\1\end{bmatrix}$.

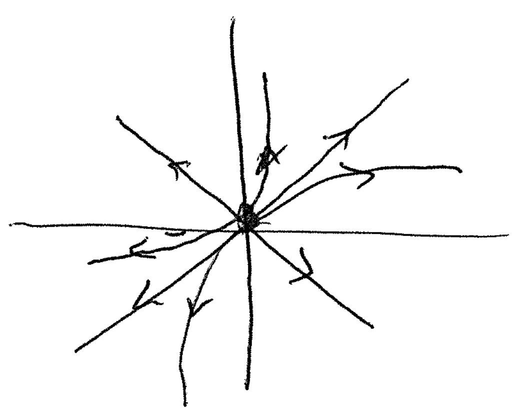


$\vec{x}(t) \to 0$ along line L_1.

This is an example of a <u>sink</u>. For a sink,

$\vec{x}(t) \to 0$ along line L_1 if λ_1 is "less negative" than λ_2.

<u>Example</u>: $c_1 e^{t}\begin{bmatrix}1\\1\end{bmatrix} + c_2 e^{2t}\begin{bmatrix}1\\-1\end{bmatrix}$



The source looks just like the sink, except time is reversed.

Lecture Notes: To Replace 7.6-7.8

Last time: to compute eigenvalues and eigenvectors for a matrix A, we first compute the characteristic equation, which is a quadratic equation:

$$\lambda^2 - \text{Tr}(A)\lambda + \det(A) = 0$$

And in the case that the discriminant is positive, we have two real distinct eigenvalues. Today we look at the other two cases.

Case 2: Complex Eigenvalues

We'll start this section with a specific example:

$$A = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix} \quad \Rightarrow \quad \begin{aligned} (3-\lambda)v_1 - 2v_2 &= 0 \\ v_1 + (1-\lambda)v_2 &= 0 \end{aligned}$$

SOLUTION: Form the characteristic equation using the shortcut or by taking the determinant of the coefficient matrix:

$$\lambda^2 - \text{Tr}(A)\lambda + \det(A) = 0 \qquad \lambda^2 - 4\lambda + 5 = 0 \qquad \lambda = 2 \pm i$$

Now, if $\lambda = 2 + i$, solve for an eigenvector:

$$\begin{array}{rrl} (3-(2+i))v_1 & -2v_2 & = 0 \\ v_1 & +(1-(2+i))v_2 & = 0 \end{array} \quad \Rightarrow \quad \begin{array}{rrl} (1-i)v_1 & -2v_2 & = 0 \\ v_1 & +(-1-i)v_2 & = 0 \end{array}$$

Using the first equation, $(1-i)v_1 - 2v_2 = 0$, our shortcut gives $\mathbf{v} = \begin{bmatrix} 2 \\ 1-i \end{bmatrix}$.

As a side remark, the other eigenvalue/eigenvector is actually the complex conjugate (we won't be using them):

$$\lambda_2 = 2 - i \qquad \mathbf{v} = \begin{bmatrix} 2 \\ 1+i \end{bmatrix}$$

The next section tells us how to solve the system.

Applying Complex evals to Systems of DEs

Suppose we have a complex eigenvalue, $\lambda = a \pm ib$. Use one of them to construct the corresponding eigenvector (complex) $\mathbf{v}$. We can then solve the system using the theorem below.

Theorem: Given complex eigenvalue λ with eigenvector $\mathbf{v}$, the solution to the system of differential equations is:

$$\mathbf{x}(t) = C_1 \text{Re}\left(e^{\lambda t}\mathbf{v}\right) + C_2 \text{Im}\left(e^{\lambda t}\mathbf{v}\right)$$

Notice that this is the extension of what we did in Chapter 3- and in fact, the justification is exactly the same. At the end of these notes, we show that the real and imaginary parts of the solution **are in fact solutions themselves**, so that we can form the general solution by taking the combination.

Below, we apply the theorem to our previous example.

Example

Give the general solution to the system $\mathbf{x}' = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix} \mathbf{x}$

This is the system for which we already have the eigenvalues and eigenvectors:

$$\lambda = 2 + i \qquad \mathbf{v} = \begin{bmatrix} 2 \\ 1-i \end{bmatrix}$$

Now, compute $e^{\lambda t}\mathbf{v}$:

$$e^{(2+i)t}\begin{bmatrix} 2 \\ 1-i \end{bmatrix} = e^{2t}(\cos(t) + i\sin(t))\begin{bmatrix} 2 \\ 1-i \end{bmatrix} =$$

$$e^{2t}\begin{bmatrix} 2\cos(t) + 2i\sin(t) \\ (\cos(t) + \sin(t)) + i(-\cos(t) + \sin(t)) \end{bmatrix}$$

so that the general solution is given by:

$$\mathbf{x}(t) = C_1 e^{2t}\begin{bmatrix} 2\cos(t) \\ \cos(t) + \sin(t) \end{bmatrix} + C_1 e^{2t}\begin{bmatrix} 2\sin(t) \\ -\cos(t) + \sin(t) \end{bmatrix}$$

Geometrically, we have two things going on. The sine and cosine functions are providing a rotation of the solution, and the exponential, e^{2t}, is expanding the solution away from the origin. Putting these two things together, we might guess that solutions in the plane are spiraling away from the origin (and in fact, that is the case). In this case, we would call the origin a *spiral source*.

Example

Give the general solution to the system: $\mathbf{x}' = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix}\mathbf{x}$

First, the characteristic equation: $\lambda^2 + 1 = 0$, so that $\lambda = \pm i$.

Now we solve for the eigenvector to $\lambda = i$:

$$\begin{aligned} (2-i)v_1 - 5v_2 &= 0 \\ 1v_1 + (-2-i)v_2 &= 0 \end{aligned}$$

Using the second equation, $v_1 - (2+i)v_2 = 0$, and we have our eigenvalue/eigenvector pair. Now we compute the needed quantity, $e^{\lambda t}\mathbf{v}$:

$$e^{it}\begin{bmatrix} 2+i \\ 1 \end{bmatrix} = (\cos(t) + i\sin(t))\begin{bmatrix} 2+i \\ 1 \end{bmatrix} = \begin{bmatrix} (\cos(t) + i\sin(t))(2+i) \\ \cos(t) + i\sin(t) \end{bmatrix}$$

Simplifying, we get:

$$\begin{bmatrix} (2\cos(t) - \sin(t)) + i(2\sin(t) + \cos(t)) \\ \cos(t) + i\sin(t) \end{bmatrix}$$

The solution is:

$$\mathbf{x}(t) = C_1\begin{bmatrix} 2\cos(t) - \sin(t) \\ \cos(t) \end{bmatrix} + C_2\begin{bmatrix} 2\sin(t) + \cos(t) \\ \sin(t) \end{bmatrix}$$

We will quickly verify that this is what we would get using the techniques of Chapter 3. From the second equation, solve for x_1, then use the first equation to get a second order DE for x_2.

$$x_1 = x_2' + 2x_2 \quad \Rightarrow \quad (x_2'' + 2x_2') = 2(x_2' + 2x_2) - 5x_2 \quad \Rightarrow \quad x_2'' + x_2 = 0$$

Therefore, $x_2 = C_1\cos(t) + C_2\sin(t)$. Solving for x_1:

$$x_1 = x_2' + 2x_2 = (-C_1\sin(t) + C_2\cos(t)) + 2(C_1\cos(t) + C_2\sin(t))$$

and we see that we get the identical solution.

Graphically, the solutions are ellipses. In fact, if we solve the differential equation by computing dy/dx, we get solutions of the form:

$$x^2 - 4xy + 5y^2 = C$$

Graphical Summary- Complex Eigenvalues

If $\lambda = \alpha + \beta i$, then $e^{\alpha t}$ determines if there is a spiral instead of a periodic solution, and determines if the solution "blows up" or converges to the origin:

- If $\alpha = 0$, we get pure periodic solutions (the period depends on β).
- If $\alpha < 0$, the origin is a *spiral sink.*
- If $\alpha > 0$, the origin is a *spiral source.*

Scratch Work

1. From our first example, show that these equations are actually multiples of each other:

$$\begin{aligned} (1-i)v_1 \quad & -2v_2 &= 0 \\ v_1 \quad & +(-1-i)v_2 &= 0 \end{aligned}$$

SOLUTION: If you divide the first equation by $1-i$, we get:

$$\frac{1-i}{1-i}v_1 - \frac{2}{1-i}v_2 = 0 \quad \Rightarrow \quad v_1 - \frac{2(1+i)}{(1^2+1^2)}v_2 = 0 \quad \Rightarrow v_1 - (1+i)v_2 = 0$$

which is the second equation.

2. Given the matrix $A = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$, with $\lambda = 2 + i$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 1-i \end{bmatrix}$, show that

$$\mathbf{z}_1(t) = \text{Re}(e^{\lambda t}\mathbf{v}) = e^{2t}\begin{bmatrix} 2\cos(t) \\ \cos(t) + \sin(t) \end{bmatrix}, \quad \mathbf{z}_2(t) = \text{Im}(e^{\lambda t}\mathbf{v}) = e^{2t}\begin{bmatrix} 2\sin(t) \\ -\cos(t) + \sin(t) \end{bmatrix}$$

each solves the system (by direct substitution).

SOLUTION: To verify that something is a solution, we want to substitute it into the differential equation to see if we have a true statement. In this example, we'll first differentiate z_1, then compute Az_1 and see if we have the same quantity.

$$\begin{bmatrix} 2e^{2t}\cos(t) \\ e^{2t}(\cos(t) + \sin(t)) \end{bmatrix}' = \begin{bmatrix} 4e^{2t}\cos(t) - 2e^{2t}\sin(t) \\ 2e^{2t}(\cos(t) + \sin(t)) + e^{2t}(-\sin(t) + \cos(t)) \end{bmatrix}$$

$$= e^{2t}\begin{bmatrix} 4\cos(t) - 2\sin(t) \\ 3\cos(t) + \sin(t) \end{bmatrix}$$

And for the matrix side of things (factor out the exponential function):

$$e^{2t}\begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2\cos(t) \\ \cos(t) + \sin(t) \end{bmatrix} = e^{2t}\begin{bmatrix} 4\cos(t) - 2\sin(t) \\ 3\cos(t) + \sin(t) \end{bmatrix}$$

A similar computation works for the imaginary part, $z_2(t)$.

Homework Set 3 (Complex Eigenvalues)

In this homework set, we will practice finding eigenvalues and eigenvectors when the eigenvalues are either complex or the matrix is defective.

1. Exercises 1, 3, pg. 409 (Section 7.6, solve with complex evals/evecs)
2. Given the eigenvalues and eigenvectors for some matrix A, write the general solution to $\mathbf{x}' = A\mathbf{x}$. Furthermore, classify the origin as a sink, source, spiral sink, spiral source, saddle, or none of the above.

   (a) $\lambda = -1 + 2i \qquad \mathbf{v} = \begin{bmatrix} 1 - i \\ 2 \end{bmatrix}$

   (b) $\lambda = -2, 3 \qquad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

   (c) $\lambda = 2, -3 \qquad \mathbf{v}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

   (d) $\lambda = 1 + 3i \qquad \mathbf{v} = \begin{bmatrix} 1 \\ 1 - i \end{bmatrix}$

   (e) $\lambda = 2i \qquad \mathbf{v} = \begin{bmatrix} 1 + i \\ 1 \end{bmatrix}$